



## Initial Value Problems

*See online and lecture notes for full details*

### §2.2: Euler's Method

MA385/530 – Numerical Analysis 1

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Our goal is to generate numerical solutions to initial value differential equations. The solutions to such problems are functions (usually, of one variable that we'll denote  $t$ ). Our approximation will give estimates of the values of this function at certain points.

We'll denote the points we are approximating at by

$$t_0 < t_1 < \cdots < t_n.$$

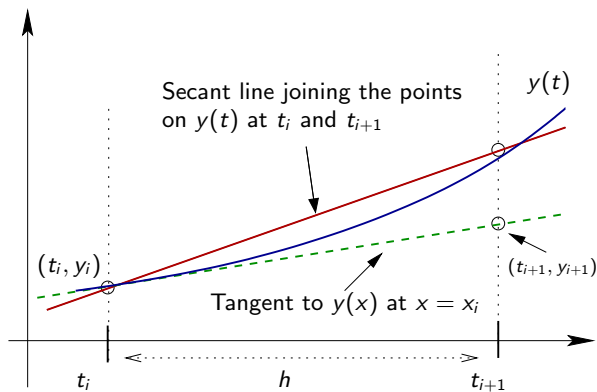
The methods we'll use are all **one-step** methods, and the first example we'll consider is **Euler's Method**.

Although it is not too important, we'll make the assumption that the points are equally spaced. So

$$t_{i+1} - t_i = \frac{t_n - t_0}{n} = h.$$

The simplest method is *Euler's Method*. Suppose we know  $y(t_i)$ , and want  $y(t_{i+1})$ . From the DE we have the slope of the tangent to  $y$  at  $t_i$ . If this approximates the slope of the line joining  $(t_i, y(t_i))$  and  $(t_{i+1}, y(t_{i+1}))$ , then

$$y'(t_i) = f(t_i, y(t_i)) \approx \frac{y_{i+1} - y_i}{t_{i+1} - t_i}.$$



**Method (Euler's Method)**

Choose equally spaced points  $t_0, t_1, \dots, t_n$  so that

$$t_i - t_{i-1} = h = (t_n - t_0)/n \quad \text{for } i = 0, \dots, n-1.$$

We call  $h$  the “time step”. Let  $y_i$  denote the approximation for  $y(t)$  at  $t = t_i$ . Set

$$y_{i+1} = y_i + hf(t_i, y_i), \quad i = 0, 1, \dots, n-1. \quad (1)$$

**Example**

Taking  $h = 1$ , estimate  $y(4)$  where

$$y'(t) = y/(1 + t^2), \quad y(0) = 1.$$

Choosing  $h = 1$  we get

■  $i = 0$ :  $t_0 = 0, y_0 = 1.$

■  $i = 1$ :  $t_1 = t_0 + h = 1.$

$$y_1 = y_0 + hf(t_0, y_0) = 1 + \frac{1}{1+0^2} = 2.$$

■  $i = 2$ :  $t_2 = t_0 + 2h = 2.$

$$y_2 = y_1 + hf(t_1, y_1) = 2 + 1 \frac{2}{1+1^2} = 3.$$

■  $i = 3$ :  $t_3 = t_0 + 3h = 3.$

$$y_3 = y_2 + hf(t_2, y_2) = 3 + 1 \frac{3}{1+2^2} = 3.6$$

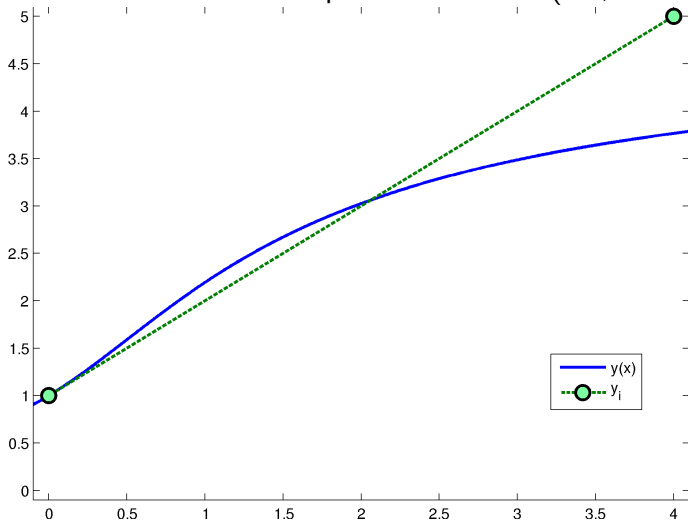
■  $i = 4$ :  $t_n = t_4 = t_0 + 4h = 4.$

$$y_n = y_4 = y_3 + hf(t_3, y_3) = 3.6 + \frac{3.6}{1+3^2} = \mathbf{3.96}$$

If we had chosen  $h = 4$  we would have only required one step:  
 $y_n = y_0 + 4f(t_0, y_0) = \mathbf{5}$ . However, this would not be very accurate.

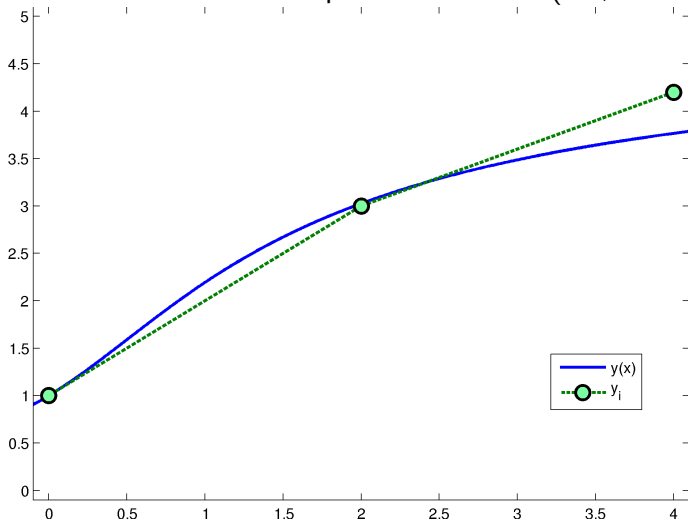
With a little work one can show that the solution to this problem is  $y(t) = e^{\tan^{-1}(t)}$  and so  $y(4) = 3.7652$ . Hence the computed solution with  $h = 1$  is much more accurate than the computed solution when  $h = 4$ . This is also demonstrated in next figure below, and in the follow table, where we see that the error seems to be proportional to  $h$ .

Euler's method for Example 1 with  $n = 1$  (i.e.,  $h = 4$ )

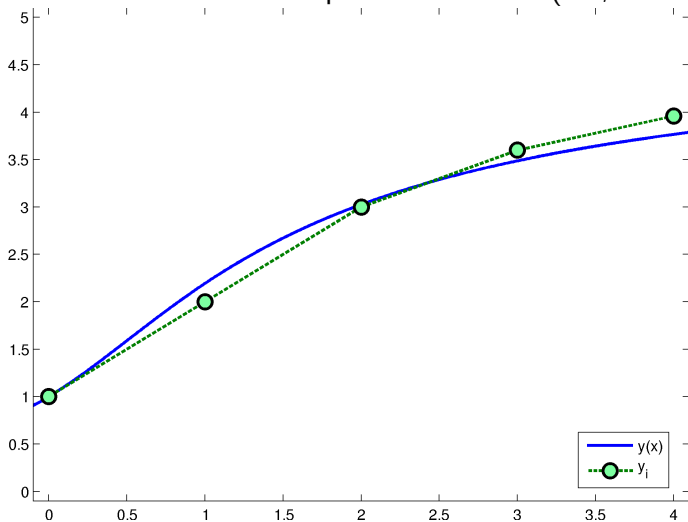




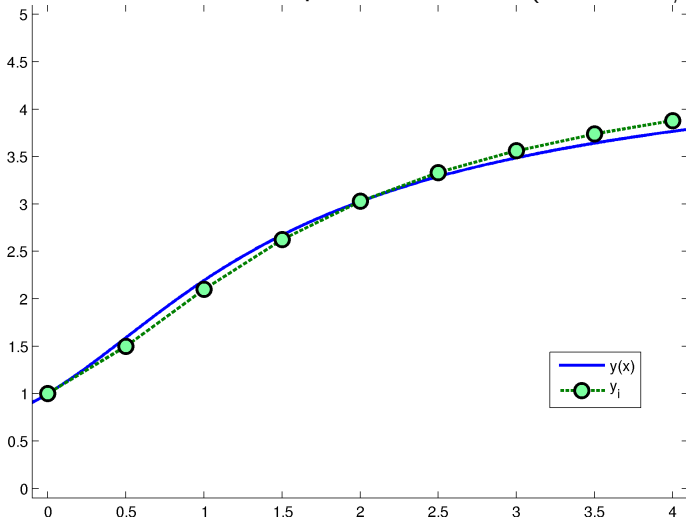
Euler's method for Example 1 with  $n = 2$  (i.e.,  $h = 2$ )



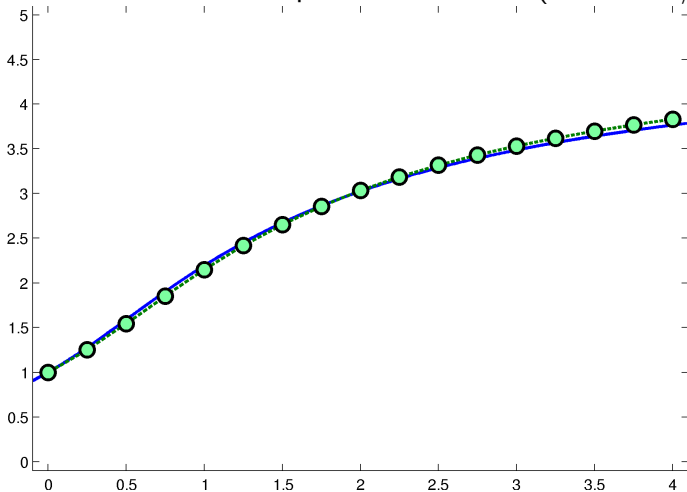
Euler's method for Example 1 with  $n = 4$  (i.e.,  $h = 1$ )



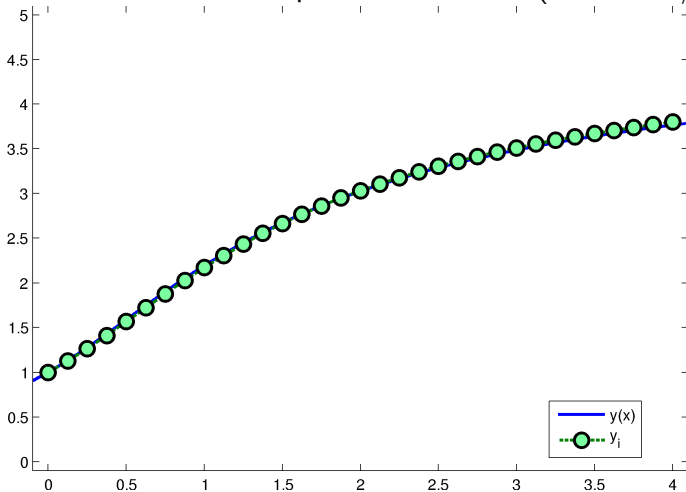
Euler's method for Example 1 with  $n = 8$  (i.e.,  $h = 1/2$ )



Euler's method for Example 1 with  $n = 16$  (i.e.,  $h = 1/4$ )



Euler's method for Example 1 with  $n = 32$  (i.e.,  $h = 1/8$ )



$n$	$h$	$y_n$	$ y(t_n) - y_n $
1	4	5.0	1.235
2	2	4.2	0.435
4	1	3.960	0.195
8	1/2	3.881	0.115
16	1/4	3.831	0.065
32	1/8	3.800	0.035

Table: Error in Euler's method for Example 1