Initial Value Problems

See online and lecture notes for full details

§2.3: Error Analysis of one-step methods

(but mainly of Euler's Method)

MA385/530 – Numerical Analysis 1

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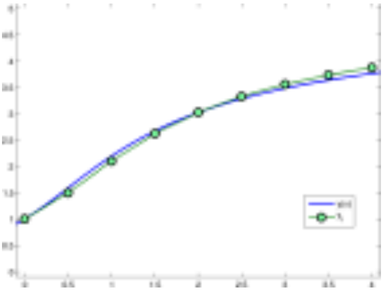

Euler's method is an example of a **one-step methods**, which have the *general* form:

$$y_{i+1} = y_i + h\Phi(t_i, y_i; h). \tag{1}$$

To get Euler's method, just take $\Phi(t_i, y_i; h) = f(t_i, y_i)$.

In the introduction, we motivated Euler's method with a geometrical argument. An alternative, more mathematical way of deriving Euler's Method is to use a *Truncated Taylor Series*:

$\Phi(t, y; h)$ is a function of t & y, but also has a parameter h.

Using a Taylor's Series:

$$y(b) = y(a) + (b-a)\,y'(a) + \tfrac{1}{2}(b-a)^2\,y''(\eta).$$

for some $\eta \in (a,b)$. Take $a = t_i$, $b = t_{i+1}$, and $h = t_{i+1} - t_i$ to get

$$y(t_{i+1}) = y(t_i) + h\,y'(t_i) + \underbrace{\frac{h^2}{2}\,y''(\eta)}_{\text{neglect this}}.$$

use $y'(t) = f(t,y)$

This not only motivates Euler's formula, but also suggests that at each step the method introduces a (local) error of $h^2 y''(\eta)/2$. (More of this later).

$$y_{i+1} = y_i + h\,f(t_i, y_i)$$

But these accumulate.....

Definition

Global Error: $\mathcal{E}_i = y(t_i) - y_i$.

True solution approximate

Definition

Truncation Error:

$$T_i := \frac{y(t_{i+1}) - y(t_i)}{h} - \Phi(t_i, y(t_i); h). \tag{2}$$

It can be helpful to think of T_i as representing how much the difference equation differs from the differential equation. We can also determine the truncation error for Euler's method directly from a Taylor Series.

ie the method.

The relationship between the global error and truncation errors is explained in the following (important!) result (also, compare with Picard's Theorem)

Theorem (Thm 12.1 in Süli & Mayers)

Let $\Phi()$ be Lipschitz with constant L. Then

$$|\mathcal{E}_n| \le T\left(\frac{e^{L(t_n-t_0)}-1}{L}\right),$$

where $T = \max_{i=0,1,\dots,n} |T_i|$.

definition (3) of T_i

Proof

$y(t_{i+1}) = y(t_i) + h\,\Phi(t_i, y(t_i); h) + h\,T_i$

$y_{i+1} = y_i + h\,\Phi(t_i, y_i; h)$

$\underbrace{y(t_{i+1}) - y_{i+1}}_{\mathcal{E}_{i+1}} = \underbrace{y(t_i) - y_i}_{\mathcal{E}_i} + h\left[\Phi(t_i, y(t_i); h) - \Phi(t_i; y_i; h)\right] + h\,T_i.$

Here we used the triangle inequality and that Φ is Lipschitz.

definition of T_i

Proof

$$y(t_{i+1}) = y(t_i) + h\,\Phi(t_i, y(t_i); h) + h\,T_i$$

$$y_{i+1} = y_i + h\,\Phi(t_i, y_i; h)$$

$$y(t_{i+1}) - y_{i+1} = y(t_i) - y_i + h\left[\Phi(t_i, y(t_i); h) - \Phi(t_i, y_i; h)\right]$$

$$|\varepsilon_{i+1}| \le |\varepsilon_i| + h\left[L\,|y(t_i) - y_i|\right] + h\,T_i$$

This gives

$$|\varepsilon_{i+1}| \le |\varepsilon_i| + h\,L\,|\varepsilon_i| + h\,T_i$$

That is

$$|\varepsilon_{i+1}| \le |\varepsilon_i|(1 + hL) + h\,T_i.$$

Since $\varepsilon_0 = 0$, we can use induction to finish the proof. (Exer).

For Euler's method, we get (from Taylor Series),

$$T = \max_{0 \le j \le n} |T_j| \le \frac{h}{2} \max_{t_0 \le t \le t_n} |y''(t)|.$$

Example

Given the problem:

$$y' = 1 + t + \frac{y}{t} \quad \textit{for } t > 1; \quad y(1) = 1,$$

find an approximation for $y(2)$.

(i) Give an upper bound for the global error taking $n = 4$ (i.e., $h = 1/4$)

(ii) What n should you take to ensure that the global error is no more that 0.1?

To answer these questions we need to use (3), which requires that we find L and an upper bound for T. In this instance, L is easy:

We are solving $y'(t) = 1 + t + y/t$, $t \in (1, 2]$

So $\Phi(t_i, y_i; h) = 1 + t_i + \dfrac{y_i}{t_i}$

Then

$$|\Phi(t_i, u; h) - \Phi(t, v; h)| \le$$

$$\left| 1 + t_i + \frac{u}{t_i} - 1 - t_i - \frac{v}{t} \right|$$

$$= \frac{1}{t_i} |u - v|$$

$$\le |u - v| \quad \text{since } 1 \le t_i \le 2 \text{ for all } i.$$

So take $L = 1$.

To find T we need an upper bound for $|y''(t)|$ on $[1, 2]$, even though we don't know $y(t)$... But we do know y',

i.e. $y' = 1 + t + y/t$. Differentiating, w.r.t. t,

to get $y'' = 0 + 1 + \dfrac{ty' - y}{t^2}$

$= 1 + \dfrac{t(1 + t + y/t) - y}{t^2}$ (using DE again)

$= 1 + \dfrac{t + t^2}{t^2} = 2 + 1/t. \leq 3$ for all t in $(1, 2]$.

Thus $T \leq \dfrac{h}{2} \max\limits_{1 \leq t \leq 2} |y''(t)| \leq \dfrac{3}{2} h.$

With these values of L and T, using (3) we find $\mathcal{E}_n \leq 0.644$. In fact, the true answer is 0.43, so we see that (3) is somewhat pessimistic.

. . . .

To answer (ii): *What n should you take to ensure that the global error is no more that 0.1?* (We should get $n = 26$. This is not that sharp: $n = 19$ will do).

We are often interested in the *convergence* of a method. That is, is it true that

$$\lim_{h\to 0} y_n = y(t_n)?$$

or $n \to \infty$, but t_n fixed.

Or equivalently that,

$$\lim_{h\to 0} \mathcal{E}_n = 0?$$

Given that the global error for Euler's method can be bounded:

$$|\mathcal{E}_n| \le h \underbrace{\frac{\max|y''(t)|}{2L}\left(\mathrm{e}^{L(t_n-t_0)}-1\right)}_{K} = hK, \tag{4}$$

we can say it converges.

So now we know, for Euler's method, that $y_n \to y(t_n)$ as $n \to \infty$, but how quickly?

Definition

The **order of accuracy** of a numerical method is p if there is a constant K so that

$$|\mathcal{E}_n| \leq Kh^p.$$

So Euler's method is first-order.

The term **order of convergence** is often use instead of **order of accuracy**.

One of the requirements for convergence is *Consistency*:

Definition

A one-step method $y_{n+1} = y_n + h\Phi(t_n, y_n; h)$ is *consistent* with the differential equation $y'(t) = f\big(t, y(t)\big)$ if $f(t, y) \equiv \Phi(t, y; 0)$.

This is clearly the case for Euler's method, since

$$\Phi(t_i, y_i; h) = f(t_i, y_i).$$

It can help to think of a method being consistent as saying tha
as h --> 0, the numerical method tends to the differential equa

Next we'll try to develop methods that are of higher order than Euler's method; that is that we can show

$$|\mathcal{E}_n| \leq Kh^p \quad \text{for some} \quad p > 1.$$

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Suppose we numerically solve some differential equation and estimate the error. If we think this error is too large we could redo the calculation with a smaller value of h. Or we could use a better method, for example *Runge-Kutta* methods. These are high-order methods that rely on evaluating $f(t, y)$ a number of times at each step in order to improve accuracy.

We'll first motivate one such method and then later look at the general framework.

The goal will be to develop some techniques to help us derive our own methods for accurately solving IVPs. Rather than using formal theory, we will reason based on carefully chosen examples.