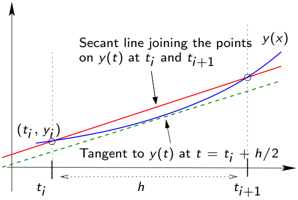
Initial Value Problems

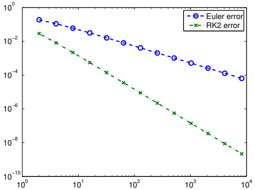
See online and lecture notes for full details

§2.4 Runge-Kutta 2

MA385/530 – Numerical Analysis 1

October 2017





Recall our original motivation of Euler's method: use the slope of the tangent to y at t_i as an approximation for the slope of the secant line joining the points $(t_i, y(t_i))$ and $(t_{i+1}, y(t_{i+1}))$.

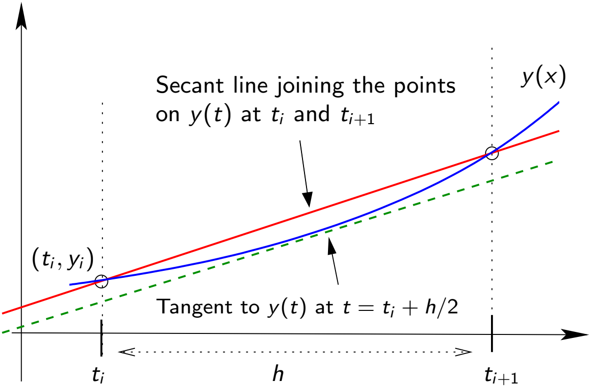
One could argue, given the diagram on the next slide, that the slope of the tangent to y at $t = (t_i + t_{i+1})/2 = t_i + h/2$ would be a better approximation. This would give

$$y(t_{i+1}) \approx y_i + hf\big(t_i + \frac{h}{2}, y(t_i + \frac{h}{2})\big). \tag{1}$$

However, we don't know $y(t_i + h/2)$, but can approximate it using Euler's Method: $y(t_i + h/2) \approx y_i + (h/2)f(t_i, y_i)$.

Substituting this into (1) gives the **Modified (Midpoint) Euler's Method**

$$y_{i+1} = y_i + hf\big(t_i + \frac{h}{2}, y_i + \frac{h}{2}f(t_i, y_i)\big). \tag{2}$$



$$y_{i+1} = y_i + hf\left(t_i + \frac{h}{2}, y_i + \frac{h}{2} f(t_i, y_i)\right).$$

Example (2.6.1)

Use the Modified Euler Method to approximate $y(1)$ *where*

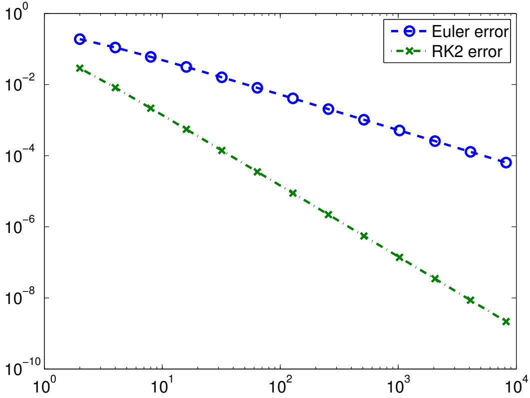
$$y(0) = 1, \qquad y'(t) = y\log(1 + t^2).$$

This has the solution $y(t) = (1 + t^2)^t \exp(-2t + 2\tan^{-1} t)$.

	Euler		Modified	
n	$\mathcal{E}_n$	$\mathcal{E}_n/\mathcal{E}_{n-1}$	$\mathcal{E}_n$	$\mathcal{E}_n/\mathcal{E}_{n-1}$
1	3.02e-01		7.89e-02	
2	1.90e-01	1.59	2.90e-02	2.72
4	1.11e-01	1.72	8.20e-03	3.54
8	6.02e-02	1.84	2.16e-03	3.79
16	3.14e-02	1.91	5.55e-04	3.90
32	1.61e-02	1.95	1.40e-04	3.95
64	8.13e-03	1.98	3.53e-05	3.98
128	4.09e-03	1.99	8.84e-06	3.99

Clearly we get a much more accurate result using the Modified Euler Method. Even more importantly, we get a higher *order of accuracy*: if h is reduced by a factor of **two**, the error in the Modified method is reduced by a factor of **four**.

We can also make a direct comparison of the two methods by using a log-log plot of the errors.



The "*Modified Euler Method*" is an example of one of the (large) family of 2^{nd}-*order Runge-Kutta* (RK2). Recalling that one-step methods are written as

$$y_{i+1} = y_i + h\Phi(t_i, y_i; h),$$

then the general RK2 method is

$$\begin{aligned} k_1 = f(t_i, y_i) \qquad & k_2 = f(t_i + \alpha h, y_i + \beta h k_1). \\ \Phi(t_i, y_i; h) &= (a k_1 + b k_2) \end{aligned} \tag{3}$$

Example 1: take $a = 1, b = 0$.
Example 2: take $\alpha = \beta = 1/2, a = 0, b = 1$.

Our aim now is to deduce general rules for choosing a, b, α and β. We'll see that if we pick any one of these four parameters, then the requirement that the method be consistent and second-order determines the other three.

By demanding that RK2 be **consistent** we get that $a + b = 1$.

Next we need to know how to choose α and β. The formal way is to use a two-dimensional Taylor series expansion. It is quite technical. Detailed notes are given in Section 2.4.5 of the notes. Instead we'll take a less rigorous, heuristic approach.

Because we expect that, for a second order accurate method, $|\mathcal{E}_n| \leq Kh^2$ where K depends on $y'''(t)$, if we choose a problem for which $y'''(t) \equiv 0$, we expect no error...

In the above example, the right-hand side of the differential equation, $f(t, y)$, depended only on t. Now we'll try the same trick: using a problem with a simple known solution (and zero error), but for which f depends explicitly on y.

Consider the DE $y(1) = 1, y'(t) = y(t)/t$. It has a simple solution: $y(t) = t$. We now use that any RK2 method should be exact for this problem to deduce that $\alpha = \beta$.

Now we collect the above results all together and show that the second-order Runge-Kutta (RK2) methods are:

$$y_{i+1} = y_i + h(ak_1 + bk_2)$$

$$k_1 = f(t_i, y_i), \qquad k_2 = f(t_i + \alpha h, y_i + \beta h k_1),$$

where we choose any $b \neq 0$ and then set

$$a = 1 - b, \qquad \alpha = \frac{1}{2b}, \qquad \beta = \alpha.$$

It is easy to verify that the Modified method satisfies these criteria.