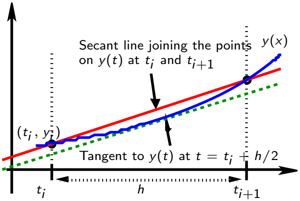
Initial Value Problems

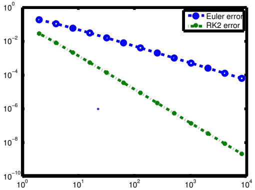
See online and lecture notes for full details

§2.4 Runge-Kutta 2

MA385/530 – Numerical Analysis 1

October 2017





Recall our original motivation of Euler's method: use the slope of the tangent to y at t_i as an approximation for the slope of the secant line joining the points $(t_i, y(t_i))$ and $(t_{i+1}, y(t_{i+1}))$.

One could argue, given the diagram on the next slide, that the slope of the tangent to y at $t = (t_i + t_{i+1})/2 = t_i + h/2$ would be a better approximation. This would give

$$y(t_{i+1}) \approx y_i + hf\big(t_i + \frac{h}{2}, y(t_i + \frac{h}{2})\big). \tag{1}$$

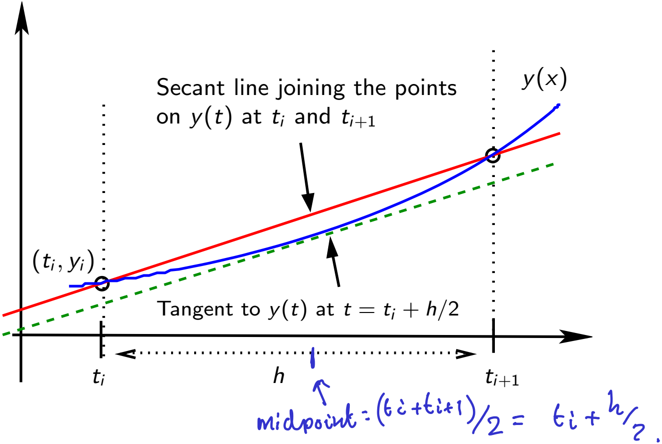
However, we don't know $y(t_i + h/2)$, but can approximate it using Euler's Method: $y(t_i + h/2) \approx y_i + (h/2)f(t_i, y_i)$.

Substituting this into (1) gives the **Modified (Midpoint) Euler's Method**

$$y_{i+1} = y_i + hf\big(t_i + \frac{h}{2}, y_i + \frac{h}{2}f(t_i, y_i)\big). \tag{2}$$

Modified Euler Method

$y'(t) = f(t, y)$



$$y_{i+1} = y_i + hf\left(t_i + \frac{h}{2}, y_i + \frac{h}{2} f(t_i, y_i)\right).$$

Example (2.6.1)

Use the Modified Euler Method to approximate $y(1)$ where

$$y(0) = 1, \qquad y'(t) = y\log(1+t^2) = f(t,y).$$

This has the solution $y(t) = (1+t^2)^t \exp(-2t + 2\tan^{-1} t)$.

We'll take $h=1$ so we just need one step: $t_0 = 0$, $t_1 = 1$.

$$y_1 = y_0 + h\, f\left(t_0 + \tfrac{h}{2},\ y_0 + \tfrac{h}{2} f(t_0, y_0)\right).$$

$$= 1 + f\left(\tfrac{1}{2},\ 1 + \tfrac{1}{2} f(0,1)\right)$$

$$= 1 + f\left(\tfrac{1}{2},\ 1 + \tfrac{1}{2}\log(1)\right) = 1 + \underbrace{f\left(\tfrac{1}{2}, 1\right)}$$

$$= 1 + \log\left(\tfrac{5}{4}\right) = 1.2231.$$

$h = 1/n$

	Euler		**Modified**	
n	$\mathcal{E}_n$	$\mathcal{E}_n/\mathcal{E}_{n-1}$	$\mathcal{E}_n$	$\mathcal{E}_n/\mathcal{E}_{n-1}$
1	3.02e-01		7.89e-02	
2	1.90e-01	1.59	2.90e-02	2.72
4	1.11e-01	1.72	8.20e-03	3.54
8	6.02e-02	1.84	2.16e-03	3.79
16	3.14e-02	1.91	5.55e-04	3.90
32	1.61e-02	1.95	1.40e-04	3.95
64	8.13e-03	1.98	3.53e-05	3.98
128	4.09e-03	1.99	8.84e-06	3.99

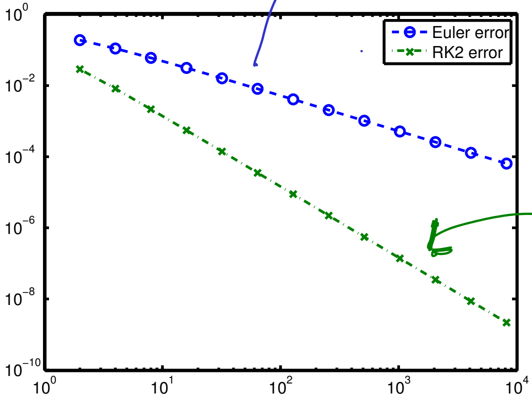
Clearly we get a much more accurate result using the Modified Euler Method. Even more importantly, we get a higher *order of accuracy*: if h is reduced by a factor of **two**, the error in the Modified method is reduced by a factor of **four**.

Modified Euler Method

We can also make a direct comparison of the two methods by using a log-log plot of the errors.

Euler's Method : slope -1

Error.



RK2 : Slope -2.

$n = 1/h$

The "*Modified Euler Method*" is an example of one of the (large) family of 2nd-*order Runge-Kutta* (RK2). Recalling that one-step methods are written as

$$y_{i+1} = y_i + h\Phi(t_i, y_i; h),$$

then the general RK2 method is

$$k_1 = f(t_i, y_i) \qquad k_2 = f(t_i + \alpha h, y_i + \beta h k_1). \qquad (3)$$
$$\Phi(t_i, y_i; h) = (ak_1 + bk_2)$$

Example 1: take $a = 1, b = 0$.
Example 2: take $\alpha = \beta = 1/2, a = 0, b = 1$.

Eg 1, If $a=1$ & $b=0$, then $\Phi(t_i, y_i; h) = K_1 = f(t_i, y_i)$

That is, we get Euler's method. Not interesting!

So we should exclude the case $a=1, b=0$.

Our aim now is to deduce general rules for choosing a, b, α and β. We'll see that if we pick any one of these four parameters, then the requirement that the method be consistent and second-order determines the other three.

By demanding that RK2 be *consistent* we get that $a + b = 1$.

The method is consistent if

$$\Phi(t_i, y_i; 0) = f(t_i, y_i).$$

when $h=0$, $K_2 = f(t_i, y_i)$.

so

$$\Phi(t_i, y_i; 0) = a f(t_i, y_i) + b f(t_i, y_i)$$
$$= (a+b) f(t_i, y_i) = f(t_i, y_i)$$

when $a+b=1$

Next we need to know how to choose α and β. The formal way is to use a two-dimensional Taylor series expansion. It is quite technical. Detailed notes are given in Section 2.4.5 of the notes. Instead we'll take a less rigorous, heuristic approach.

Because we expect that, for a second order accurate method, $|\mathcal{E}_n| \leq Kh^2$ where K depends on $y'''(t)$, if we choose a problem for which $y'''(t) \equiv 0$, we expect no error...

If $y(t) = t^2$. Then $y'(t) = 2t$, $y''(t) = 2$ & $y'''(t) \equiv 0$. Also $y(t) = t^2$ solves $y'(t) = 2t$, $y(1) = 1$.

Lets take $h = 1$. So $t_0 = 1$ & $t_1 = 2$.

Then, since $f(t, y) = 2t$. So

$$K_1 = f(t_0, y_0) = 2. \quad K_2 = f(t_0 + \alpha, y_0 + \beta K_1) = 2(t_0 + \alpha).$$

This gives

$$y_1 = y_0 + ((a)(2) + (1-a)\,2(1+\alpha)) = 3 + (1-a)2\alpha.$$

Then, since there is no error, $y_1 = 4$. So $\alpha = \frac{1}{2(1-a)} = \frac{1}{2b}$

(Finished here 12/10/17)

In the above example, the right-hand side of the differential equation, $f(t, y)$, depended only on t. Now we'll try the same trick: using a problem with a simple known solution (and zero error), but for which f depends explicitly on y.

Consider the DE $y(1) = 1, y'(t) = y(t)/t$. It has a simple solution: $y(t) = t$. We now use that any RK2 method should be exact for this problem to deduce that $\alpha = \beta$.

Here $y'(t) = \frac{y(t)}{t} = \frac{t}{t} = 1$ ✓. Lets take $h = 1$.

Next $K_1 = f(t_i, y_i)$, $K_2 = f(t_i + \alpha h, y_i + \beta h K_1)$.

Take $i = 0$, $K_1 = f(t_0, y_0) = \frac{y_0}{t_0} = \frac{1}{1} = 1$.

$K_2 = \frac{y_0 + \beta K_1}{t_0 + \alpha} = \frac{1+\beta}{1+\alpha}$. Then

$y_1 = y_0 + h(aK_1 + bK_2) = 1 + a + b\frac{1+\beta}{1+\alpha} = 2$.

Now using that $a = 1-b$, we get

$$1 + (1-b) + b\,\frac{1+\beta}{1+\alpha} = 2.$$

Rearranging, we have $b = b\,\frac{1+\beta}{1+\alpha}$.

This is true if $b=0$, but that is not possible since $\alpha = 1/2b$ would not be defined.

So instead we must have $\frac{1+\beta}{1+\alpha} = 1$.

That is

$$\alpha = \beta.$$

Now we collect the above results all together and show that the second-order Runge-Kutta (RK2) methods are:

$$y_{i+1} = y_i + h(ak_1 + bk_2)$$

$$k_1 = f(t_i, y_i), k_2 = f(t_i + \alpha h, y_i + \beta h k_1),$$

where we choose any $b \neq 0$ and then set

$$a = 1 - b, \qquad \alpha = \frac{1}{2b}, \qquad \beta = \alpha.$$

It is easy to verify that the Modified method satisfies these criteria.