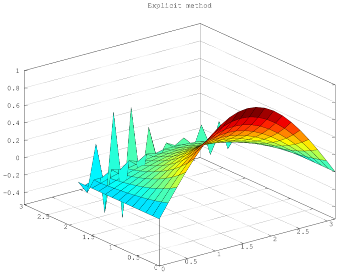
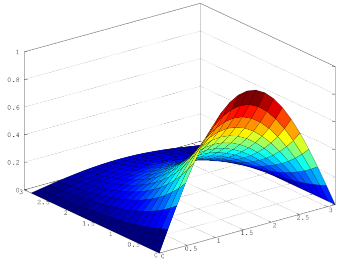




Initial Value Problems

See online and lecture notes for full details

§2.7 From IVPs to linear systems

MA385 – Numerical Analysis 1

October 2017

In this final section, we highlight some of the many important aspects of the numerical solution of IVPs that are *not* covered in detail in this course:

- Systems of ODEs;
- Higher-order equations;
- Implicit methods; and
- Problems in two dimensions.

We have the additional goal of seeing how these methods related to the earlier section of the course (nonlinear problems) and next section (linear equation solving).

So far we have solved only single IVPs. However, must interesting problems are coupled systems: find functions y and z such that

$$y'(t) = f_1(t, y, z),$$

$$z'(t) = f_2(t, y, z).$$

This does not present much of a problem to us. For example the Euler Method is extended to

$$y_{i+1} = y_i + hf_1(t, y_i, z_i),$$

$$z_{i+1} = z_i + hf_2(t, y_i, z_i).$$

Example

In pharmokinetics, the flow of drugs between the blood and major organs can be modelled

$$\frac{dy}{dt}(t) = k_{21}z(t) - (k_{12} + k_{\mathrm{elim}})y(t).$$
$$\frac{dz}{dt}(t) = k_{12}y(t) - k_{21}z(t).$$
$$y(0) = d, \quad z(0) = 0.$$

where y is the concentration of a given drug in the blood-stream and z is its concentration in another organ. The parameters k_{21}, k_{12} and k_{elim} are determined from physical experiments.

Example

$$\frac{dy}{dt}(t) = k_{21}z(t) - (k_{12} + k_{\mathrm{elim}})y(t).$$
$$\frac{dz}{dt}(t) = k_{12}y(t) - k_{21}z(t).$$
$$y(0) = d, \quad z(0) = 0.$$

Euler's method for this is:

$$y_{i+1} = y_i + h\big(- (k_{12} + k_{\mathrm{elim}})y_i + k_{21}z_i\big),$$
$$z_{i+1} = z_i + h\big(k_{12}y_i + k_{21}z_i\big).$$

So far we've only considered first-order initial value problems. However, the methods can easily be extended to high-order problems:

$$y''(t) + a(t)y'(t) = f(t, y); \quad y(t_0) = y_0, y'(t_0) = y_1.$$

We do this by converting the problem to a system: set $z(t) = y'(t)$. Then:

$$\begin{aligned} z'(t) &= -a(t)z(t) + f(t, y), \qquad & z(t_0) &= y_1, \\ y'(t) &= z(t), & y(t_0) &= y_0. \end{aligned}$$

Example

Consider the following 2nd-order IVP

$$y''(t) - 3y'(t) + 2y(t) + e^t = 0,$$

$$y(1) = e, \; y'(1) = 2e.$$

Although we won't dwell on the point, there are many problems for which the one-step methods we have seen will give a useful solution only when the step size, h, is small enough. For larger h, the solution can be very unstable.

Such problems are called "stiff" problems. They can be solved, but are best done with so-called "implicit methods", the simplest of which is the Implicit Euler Method:

$$y_{i+1} = y_i + hf(t_{i+1}, y_{i+1}).$$

Note that y_{i+1} appears on both sizes of the equation. To implement this method, we need to be able to solve this non-linear problem. The most common method for doing this is Newton's method.

Towards Partial Differential Equations

So far, in MA385/530, we've only considered *ordinary* differential equations: these are DEs which involve functions of just one variable. In our examples above, this variable was time.

However, many physical phenomena vary in space and time, and so the solutions to the differential equations the model them depend on two or more variables. The derivatives expressed in the equations are *partial derivatives* and so they are called *partial differential equations* (PDEs).

We will take a brief look at how to solve these (and how not to solve them). This will motivate the following section, on solving systems of linear equations.

Recall (again) the Black-Scholes equations for pricing an option:

$$\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV = 0.$$

With a little effort, (see, e.g., Chapter 5 of "*The Mathematics of Financial Derivatives: a student introduction*", by Wilmott et al.) this can be transformed to the simpler-looking *heat equation*:

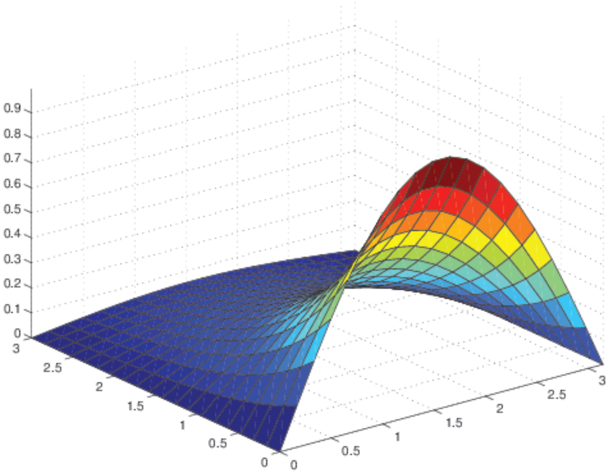
$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x), \quad \text{for} \quad (x,t) \in [0,L] \times [0,T],$$

and with the initial and boundary conditions

$$u(0,x) = g(x) \qquad \text{and } u(t,0) = a(t), u(t,L) = b(t).$$

Example

If $L = \pi$, $g(x) = \sin(x)$, $a(t) = b(t) \equiv 0$ then $u(t,x) = e^{-t}\sin(x)$.



In general, however, this problem can't be solved explicitly for arbitrary g, a, b, and so a numerical scheme is used. Suppose we somehow know $\partial^2 u/\partial x^2$, then we could just use Euler's method:

$$u(t_{i+1}, x) = u(t_i, x) + h\frac{\partial^2 u}{\partial x^2}(t_i, x).$$

Although we don't know $\frac{\partial^2 u}{\partial x^2}(t_i, x)$ we can *approximate* it:

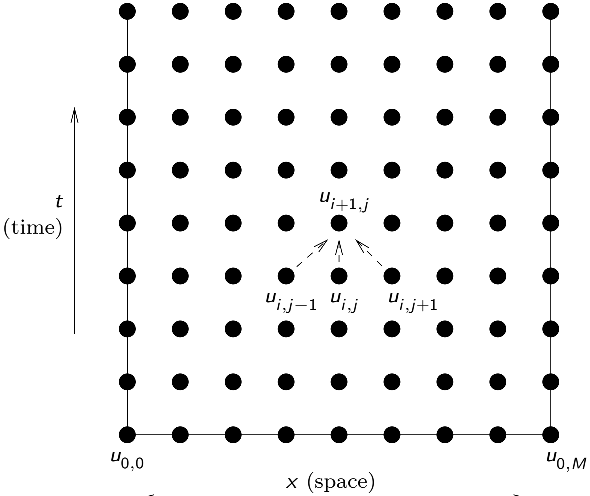
1. Divide $[0, T]$ into N intervals of width h, giving the grid $\{0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T\}$, with $t_i = t_0 + ih$.
2. Divide $[0, L]$ into M intervals of width H, giving the grid $\{0 = x_0 < x_1 < \cdots < x_M = L\}$ with $x_j = x_0 + jH$.
3. Denote by $u_{i,j}$ the approximation for $u(t, x)$ at (t_i, x_j).
4. For each $i = 0, 1, \ldots, N-1$, use the approximation:

$$\frac{\partial^2 u}{\partial x^2}(t_i, x_j) \approx \delta_x^2 u_{i,j} = \frac{1}{H^2}\big(u_{i,j-1} - 2u_{i,j} + u_{i,j+1}\big),$$

for $k = 1, 2, \ldots, M-1$.

Now set: $u_{i+1,j} := u_{i,j} - h\left[\delta_x^2 u_{i,j}\right].$

This scheme is called an **explicit method**: if we know $u_{i,j-1}$, $u_{i,j}$ and $u_{i,j+1}$ then we can explicitly calculate $u_{i+1,j}$.



Unfortunately, this method is not very stable: huge errors occur in the approximation. (Example).

Instead one might use an **implicit method**: if we know $u_{i-1,j}$, we compute $u_{i,j-1}$, $u_{i,j}$ and $u_{i,j+1}$ simultaneously:

$$u_{i,j} - h\left[\delta_x^2 u_{i,j}\right] = u_{i-1,j}$$

This is actually a set of simultaneous equations:

$$\begin{gathered} u_{i,0} = a(t_i), \\ \alpha u_{i,j-1} + \beta u_{i,j} + \alpha u_{i,j+1} = u_{i-1,k}, \quad k = 1, 2, \ldots, M-1 \\ u_{i,M} = b(t_i), \end{gathered}$$

where $\alpha = -\frac{h}{H^2}$ and $\beta = \frac{2h}{H^2} + 1$.

This could be expressed more clearly as the matrix-vector equation:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \alpha & \beta & \alpha & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & \alpha & \beta & \alpha & \dots & 0 & 0 & 0 & 0 \\ & \vdots & & & \ddots & & & \vdots & \\ 0 & 0 & 0 & 0 & \dots & \alpha & \beta & \alpha & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \alpha & \beta & \alpha \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_{i,0} \\ u_{i,1} \\ u_{i,2} \\ \vdots \\ u_{i,n-2} \\ u_{i,n-1} \\ u_{i,n} \end{pmatrix} = \begin{pmatrix} a(0) \\ u_{i-1,1} \\ u_{i-1,2} \\ \vdots \\ u_{i-1,n-2} \\ u_{i-1,n-1} \\ b(T) \end{pmatrix}.$$

So "all" we have to do now is solve this system of equations. That is what the next section of the course is about.