Exer 1.1

No – it is not necessary that $f(a)f(b) \leq 0$ for there to be a solution to $f(x)=0$ in the interval $[a,b]$.

For example, suppose that $f(x) = x^2$, $a=-1$, and $b=1$.

Then there is a solution to $f(x)=0$, in $[-1,1]$, i.e., at $x=0$, even though $f(-1)f(1) = 1 \not\leq 0$.

Exer 1.2

We know that $|\tau - x_k| \leq \left(\frac{1}{2}\right)^{k-1} |b-a|$.

So, if $\left(\frac{1}{2}\right)^{k-1} |b-a| \leq \varepsilon$, then $|\tau - x_k| \leq \varepsilon$ too.

Therefore, we need $k$ such that $\left(\frac{1}{2}\right)^{k-1} \leq \frac{\varepsilon}{|b-a|}$

If we express this as $2^{k-1} \geq |b-a|/\varepsilon$, then we need $k-1 \geq \log_2 (|b-a|/\varepsilon)$.

It follows that, if $K = \lceil \log_2 (|b-a|/\varepsilon) \rceil + 1$ then $|\tau - x_k| \leq \varepsilon$ for all $k \geq K$.

This estimate is entirely independent of $f$ (which was not the case with the Secant Method, or Newton's Method).

It does depend on $a$ & $b$: if $|b-a|$ were doubled, one extra iteration would be required.

Exer 1.5 The equation is $y - f(x_k) = \left[\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}\right](x - x_k)$

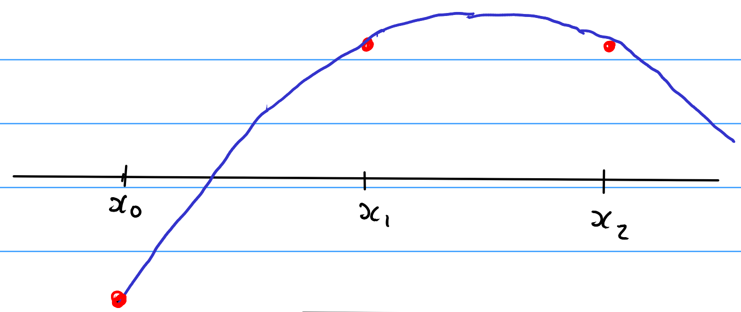
We take the zero of this line to be $x_{k+1}$. That is

$$0 - f(x_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}(x_{k+1} - x_k)$$

Rearrange to get $x_{k+1} = x_k - f(x_k) \cdot \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}$.

Exer 1.6

(i) Yes. One way to do this is to choose a function $f$ & interval $[a,b]$ such that $f$ is continuous, $f(a) < 0$, $f(b) > 0$ but $f(b) = f\left(\frac{a+b}{2}\right)$. Then bisection will work, but we'll find that $f(x_1) = f(x_2)$ so applying the Secant Method will lead to dividing by zero.




Eg try $f(x) = -x^2 + 3x - 1$ with $a = x_0 = 0$, $b = x_1 = 1$.

Exer 1.6 (ii) This problem is really an exercise in articulating the assumptions that we are making regarding the function f. Further, there are two possible correct answers, providing you explain your reasoning correctly.

(a) If we assume that the functon f is continuous and defined on [a,b], and furthermore that f(a)f(b)≤0, then bisection must converge. So there is no such case where bisection will fail, but the secant succeed.

(b) Suppose that f(x)=0 has a solution in [a,b], but f(a)f(b)>0. In that case, bisection cannot even begin, but the secant method may work. For example, take

$$f(x) = x^2 - 2x + 1$$ on [-1,2].

This has a (double) root at x=1, but f(-1)f(2)=(4)(1)>0. So we can't apply bisection. But one can verify that the Secant method will converge (though slowly).

Exer 1.7

The equation of the line through the point $(x_k, f(x_k))$, with slope $f'(x_k)$ is

$$y - f(x_k) = f'(x_k)(x - x_k).$$

Now take $x_{k+1}$ to be the point where this line is zero. That is,

$$0 - f(x_k) = f'(x_k)(x_{k+1} - x_k).$$

Rearranging we get $x_{k+1} = x_k - f(x_k)/f'(x_k)$.

Exer 1.9

The Newton Error Formula is

$$\tau - x_{k+1} = -\frac{(\tau - x_k)^2}{2}\,\frac{f''(\eta_k)}{f'(x_k)}.$$

In this case, given the bounds on $f''$ & $f'$ we get

$$|\tau - x_{k+1}| \le \frac{3}{2}(\tau - x_k)^2.$$

Since $|\tau - x_0| \le \frac{1}{2}$, we know $|\tau - x_1| \le \frac{3}{8}$.

Similarly $|\tau - x_2| \le \frac{3}{2}\left(\frac{3}{8}\right)^2 = \frac{27}{128} \sim 0.21094$

And $|\tau - x_3| \le \left(\frac{3}{2}\right)\left(\frac{27}{128}\right)^2 \sim 0.0667$.

Exer 1.11

(a) This method is sometimes called "the discrete Newton Method." First we write it as

$$x_{k+1} = x_k - f(x_k)\Big/\left[\frac{f(x_k + f(x_k)) - f(x_k)}{f(x_k)}\right]$$

Suppose it converges, then $\lim_{k\to\infty} x_k = \tau$ so $\lim_{k\to\infty} f(x_k) = 0$.

Recalling that

$$\lim_{\delta\to 0}\frac{f(x+\delta) - f(x)}{\delta} = f'(x),$$ we see that

$$\lim_{k\to\infty}\left[\frac{f(x_k + f(x_k)) - f(x_k)}{f(x_k)}\right] = f'(x_k).$$

So, as $k\to\infty$, the method converges to Newton's Method.

Exer 1.12 (i) f has a double root at $\tau$.

(ii) we have that $\tau - x_{k+1} = -\frac{1}{2}(\tau - x_k)^2 \frac{f''(\eta_k)}{f'(x_k)}$ (Newton Error Formula)

Using the Mean Value Theorem, there is a point $\mu_k \in [x_k, \eta_k]$ such that

$$\frac{f'(\tau) - f'(x_k)}{\tau - x_k} = f''(\mu_k).$$

Since $f'(\tau) = 0$, we can substitute this into the Newton Error Formula to get

$$\tau - x_{k+1} = \frac{1}{2}(\tau - x_k) \cdot \frac{f''(\eta_k)}{f''(\mu_k)}$$

as required.

(iii) This formula tells us that Newton's method will converge more slowly in this case: the rate of convergence is (at least) linear, instead of quadratic.

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Exer 1.15

(i) Need to find points s.t.

$$\frac{x^2}{4} + \frac{5}{4}x - \frac{1}{2} = x \quad \Rightarrow \quad x^2 + x - 2 = 0$$

This has roof $x = -2$ & $x = 1$.

(ii) If $|g'(x)| < 1$ on a region then, by the mean value thm, $g$ is a contraction.

Here $g'(x) = \frac{x}{2} + \frac{5}{4}$.

$$\frac{x}{2} + \frac{5}{4} > -1 \Rightarrow \frac{x}{2} > -\frac{9}{4} \Rightarrow x > -\frac{9}{2}.$$

$$\frac{x}{2} + \frac{5}{4} < 1 \Rightarrow \frac{x}{2} < -\frac{1}{4} \Rightarrow x < -\frac{1}{2}$$

So $g$ is a contraction in the region $[-\frac{9}{2}, -\frac{1}{2}]$ about $x = -2$, and not at all about $x = 1$.

1.15



(i) Write out the Taylor series for $g(x_{k+1})$ about $x = \tau$, truncating at the $\mathcal{O}(g^{(p)})$ term:

$$g(x_k) = g(\tau) + (x_k - \tau) g'(\tau) + \tfrac{1}{2}(x_k - \tau)^2 g''(\tau) + \dots$$
$$+ \frac{1}{(p-1)!}(x_k - \tau)^{p-1} g^{(p-1)}(\tau) + \frac{1}{p!}(x_k - \tau)^p g^{(p)}(\eta) \quad \text{some } \eta \in [x_k, \tau]$$

Using that $g(x_k) = x_{k+1}$, $g(\tau) = \tau$ and $g'(\tau) = g''(\tau) = \dots = g^{(p-1)}(\tau) = 0$ we get

$$x_{k+1} = \tau + (x_k - \tau)^p \frac{g^{(p)}(\eta)}{p!}$$

So $$\frac{x_{k+1} - \tau}{(x_k - \tau)^p} = \frac{g^{(p)}(\eta)}{p!}$$

We are told that the method converges, so $\lim_{k\to\infty} x_k = \tau$. Since $\eta \in [x_k, \tau]$, it follows that $\lim_{k\to\infty} \eta = \tau$.

Thus $\lim_{k\to\infty} \frac{x_k - \tau}{(x_k - \tau)^p} = \mu$, where $\mu$ is the constant $\frac{g^{(p)}(\tau)}{p!}$

(ii) If we write Newton's Method as $x_{k+1} = g(x_k)$ with $g(x) = x - f(x)/f'(x)$, to show that this is of order 2, we need (a) $g(\tau) = \tau$ (b) $g'(\tau) = 0$.

For (a), use that $f(\tau) = 0$, to get $g(\tau) = \tau - 0/f'(\tau) = \tau$ since $f'(\tau) \neq 0$.

For (b), differentiate $g$ to get $$g'(x) = 1 - \frac{(f'(x))^2 - f(x) f''(x)}{(f'(x))^2}$$
$$= \frac{f(x) f''(x)}{(f'(x))^2}$$

Now use $f(\tau) = 0$ to get $g'(\tau) = 0$.

Therefore, we can apply Part (i) with $p = 2$ to get the desired result.