Chapter 1: Solving nonlinear equations

The following are homework exercises: Exercises 1.5. 1.6, 1.7, 1.11(a) and 1.12

Exercise 1.1. Does Proposition 1.1.1 mean that, if there is a solution to $f(x) = 0$ in $[a, b]$ then $f(a)f(b) \leqslant 0$? That is, is $f(a)f(b) \leqslant 0$ a *necessary* condition for their being a solution to $f(x) = 0$? Give an example that supports your answer.

Exercise 1.2. Suppose we want to find $\tau \in [a, b]$ such that $f(\tau) = 0$ for some given f, a and b. Write down an estimate for the number of iterations K required by the bisection method to ensure that, for a given ε, we know $|x_k - \tau| \leqslant \varepsilon$ for all $k \geqslant K$. In particular, how does this estimate depend on f, a and b?

Exercise 1.3. How many (decimal) digits of accuracy are gained at each step of the bisection method? (If you prefer, how many steps are need to gain a single (decimal) digit of accuracy?)

Exercise 1.4. Let $f(x) = e^x - 2x - 2$. Show that there is a solution to the problem: *find* $\tau \in [0, 2]$ *such that* $f(\tau) = 0$. Taking $x_0 = 0$ and $x_1 = 2$, use 6 steps of the bisection method to estimate τ. Give an upper bound for the error $|\tau - x_6|$. (You may use a computer program to do this).

Exercise 1.5. $\star$ Suppose we define the Secant Method as follows.

> *Choose any two points* x_0 *and* x_1.
> *For* $k = 1, 2, \ldots$, *set* x_{k+1} *to be the point where the line through* $\big(x_{k-1}, f(x_{k-1})\big)$ *and* $\big(x_k, f(x_k)\big)$ *that intersects the* x*-axis.*

Show how to derive the formula for the secant method.

Exercise 1.6. $\star$

(i) Is it possible to construct a problem for which the bisection method will work, but the secant method will fail? If so, give an example.

(ii) Is it possible to construct a problem for which the secant method will work, but bisection will fail? If so, give an example.

Exercise 1.7. $\star$ Write down the equation of the line that is tangential to the function f at the point x_k. Give an expression for its zero. Hence show how to derive Newton's method.

Exercise 1.8. (i) It is possible to construct a problem for which the bisection method will work, but Newton's method will fail? If so, give an example.

(ii) It is possible to construct a problem for which Newton's method will work, but bisection will fail? If so, give an example.

Exercise 1.9. (i) Write down Newton's Method as applied to the function $f(x) = x^3 - 2$. Simplify the computation as much as possible. What is achieved if we find the root of this function?

(ii) Do three iterations by hand of Newton's Method applied to $f(x) = x^3 - 2$ with $x_0 = 1$.

Exercise 1.10. (This is taken from Exercise 3.5.1 of Epperson). If f is such that $|f''(x)| \leqslant 3$ and $|f'(x)| \geqslant 1$ for all x, and if the initial error in Newton's Method is less than $1/2$, give an upper bound for the error at each of the first 3 steps.

Exercise 1.11. Here is (yet) another scheme called *Steffenson's Method*: Choose $x_0 \in [a, b]$ and set

$$x_{k+1} = x_k - \frac{\big(f(x_k)\big)^2}{f\big(x_k + f(x_k)\big) - f(x_k)} \quad \text{for } k = 0, 1, 2, \ldots.$$

(a) $\star$ Explain how this method relates to Newton's Method.

(b) [Optional] Write a program, in MATLAB, or your language of choice, to implement this method. Verify it works by using it to estimate the solution to $e^x = (2 - x)^3$ with $x_0 = 0$. Submit your code and test harness as Blackboard assignment. *No credit is available for this part, but feedback will be given on your code. Also, it will help you prepare for the final exam.*

Exercise 1.12. $\star$ (This is Exercise 1.6 from Süli and Mayers) The proof of the convergence of Newton's method given in Theorem 1.3.5 uses that $f'(\tau) \neq 0$. Suppose that it is the case that $f'(\tau) = 0$.

(i) What can we say about the root, τ?

(ii) Starting from the Newton Error formula, show that

$$\tau - x_{k+1} = \frac{(\tau - x_k)}{2} \frac{f''(\eta_k)}{f''(\mu_k)},$$

for some μ_k between τ and x_k. (*Hint: try using the MVT*).

(iii) What does the above error formula tell us about the convergence of Newton's method in this case?

Exercise 1.13. Is it possible for g to be a contraction on $[a, b]$ but not have a fixed point in $[a, b]$? Give an example to support your answer.

Exercise 1.14. Show that $g(x) = \ln(2x + 1)$ is a contraction on $[1, 2]$. Give an estimate for L. (Hint: Use the Mean Value Theorem).

Exercise 1.15. Consider the function $g(x) = x^2/4 + 5x/4 - 1/2$.

(i) It has two fixed points – what are they?

(ii) For each of these, find the largest region around them such that g is a contraction on that region.

Exercise 1.16. Although we didn't prove it in class, it turns out that, if $g(\tau) = \tau$, and the fixed point method given by

$$x_{k+1} = g(x_k),$$

converges to the point τ (where $g(\tau) = \tau$), and

$$g'(\tau) = g''(\tau) = \cdots = g^{(p-1)}(\tau) = 0,$$

then it converges with order p.

(i) Use a Taylor Series expansion to prove this.

(ii) We can think of Newton's Method for the problem $f(x) = 0$ as fixed point iteration with $g(x) = x - f(x)/f'(x)$. Use this, and Part (i), to show that, if Newton's method converges, it does so with order 2, providing that $f'(\tau) \neq 0$.