2.5 LAB 2: Euler's Method

In this session you'll develop your knowledge of MATLAB by using it to implement Euler's Method for IVPs, and study its their order of accuracy.

You should be able to complete at least Section 2.5.8 today. At the end of the class, **verify your participation by uploading your results to Blackboard** (go to "Assignments and Labs" and then "Lab 2"). In Lab 3, you'll implement higher-order schemes.

2.5.1 Four ways to define a vector

We know that the most fundamental object in MATLAB is a matrix. The the simplest (nontriveal) example of a matrix is a vector. So we need to know how to define vectors. Here are several ways we could define the vector

$$x = (0, .2, .4, \dots, 1.8, 2.0). \qquad (2.5.1)$$

```
x = 0:0.2:2 % From 0 to  2 in steps of 0.2
x = linspace(0, 2, 11); % 11 equally spaced
                  % points with x(1)=0, x(11)=2.
x = [0, 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4, ...
  1.6, 1.8, 2.0]; % Define points individually
```

The last way is rather tedious, but this one is worse:

```
x(1)=0.0; x(2)=0.2, x(3)=0.4; x(4)=0.6; ...
```

We'll see a less tedious way of doing this last approach in Section 2.5.3 below.

2.5.2 Script files

MATLAB is an *interpretative* environment: if you type a (correct) line of MATLAB code, and hit return, it will execute it immediately. For example: try `>> exp(1)` to get a decimal approximation of e.

However, we usually want to string together a collection of MATLAB operations and run the repeatedly. To do that, it is best to store these commands in a *script* file. This is done be making a file called, for example, *Lab2.m* and placing in it a series of MATLAB commands. A script file is run from the MATLAB command window by typing the name of the script, e.g., `>> Lab2`

Try putting some of the above commands for defining a vector into a script file and run it.

2.5.3 `for`–loops

When we want to run a particular set of operations a fixed number of times we use a `for`–loop.

It works by iterating over a vector; at each iteration the *iterand* takes each value stored in the vector in turn.

For example, here is another way to define the vector in (2.5.1):

```
for i=0:10 % 0:10 is the vector [0,1,...,10]
  x(i+1) = i*0.2;
end
```

2.5.4 Functions

In MATLAB we can define a function in a way that is quite similar to the mathematical definition of a function. The syntax is `>> Name = @(Var)(Formula);`
Examples:

```
f = @(x)(exp(x) - 2*x -1);
g = @(x)(log(2*x +1));
```

Now, for example, if we call $g(1)$, it evaluates as $\log(3) = 1.098612288668110$. Furthermore, if `x` is a vector, so too is `g(x)`.

A more interesting example would be to try

```
>> xk = 1;
```

and the repeat the line

```
>> xk = g(xk)
```

Try this and observe that the values of `xk` seem to be converging. This is because we are using *Fixed Point Iteration*.

Later we'll need to know how to define functions of two variables. This can be done as:
`F = @(y,t)(y./(1 + t.^2));`

2.5.5 Plotting functions

MATLAB has two ways to plot functions. The easiest way is to use a function called `ezplot`:

```
 ezplot(f, [0, 2]);
```

plots the function $f(x)$ for $0 \leqslant x \leqslant 2$. A more flexible approach is to use the `plot` function, which can plot one vector as a function of another. Try these examples below, making sure you first have defined the vector x and the functions f and g:

```
figure(1);
plot(x,f(x));
figure(2);
plot(x,f(x), x, g(x),  '--', x,x, '-.');
```

Can you work out what the syntax `'--'` and `'-.'` does? If not, ask a tutor. Also try

```
plot(x,f(x), 'g-o', x, g(x),  'r--x', ...
   x,x, '-.');
```

2.5.6 How to learn more

These notes are not an encyclopedic guide to MATLAB – they have just enough information to get started. There are many good references online.

Exercise: access Learning MATLAB by Tobin Driscoll through the NUI Galway library portal. Read Section 1.6: "Things about MATLAB that are very nice to know, but which often do not come to the attention of beginners".

2.5.7 Initial Value Problems

The particular example of an IVP that we'll look at in this lab is: *estimate* $y(4)$ *given that* is one that we had earlier in (2.2.2):

$$y'(t) = y/(1+t^2), \text{ for } t > 0, \quad \text{and } y(0) = 1.$$

The true solution to this is $y(t) = e^{\arctan(t)}$. If you don't want to solve this problem by hand, you could use Maple. The Maple command is:

```
dsolve({D(y)(t)=y(t)/(1+t^2),y(0)=1},y(t));
```

2.5.8 Euler's Method

Euler's Method is

- Choose n, the number of points at which you will estimate $y(t)$. Let $h = (t_n - t_0)/n$, and $t_i = t_0 + ih$.
- For $i = 0, 1, 2, \dots, n-1$ set $y_{i+1} = y_i + hf(t_i, y_i)$.

Then $y(t_n) \approx y_n$. As shown in Section 2.3.3, the global error for Euler's method can be bounded:

$$|\mathcal{E}_n| := |y(T) - y_n| \leqslant Kh,$$

for some constant K that does not depend on h (or n). That is, if h is halved (i.e., n is doubled), the error is halved as well.

Download the MATLAB script file `Euler.m`. It can be run in MATLAB simply by typing `>> Euler`
It implements Euler's method for $n = 1$. Read the file carefully and make sure you understand it.

The program computes a vector y that contains the estimates for y at the time-values specified in the vector t. However, MATLAB indexes all vectors from 1, and not 0. So $t(1) = t_0$, $t(2) = t_1$, ... $t(n+1) = t_n$.

By changing the value of n, complete the table.

We want to use this table to verify that Euler's Method is 1^{st}-order accurate. That is:

$$|\mathcal{E}_n| \leqslant Kh^\rho \qquad \text{with} \qquad \rho = 1.$$

A computational technique that verifies the order of the method is to estimate ρ by

$$\rho \approx \log_2\Big(\frac{|\mathcal{E}_n|}{|\mathcal{E}_{2n}|}\Big). \tag{2.5.2}$$

Table 2.3: Complete this table showing the convergence of Euler's method

n	y_n	$\mathcal{E}_n$	ρ
2	4.2	4.347×10^{-1}	
4	3.96	1.947×10^{-1}	
8			
16			
32			
64			
128			
256			
512			

Use the data in the table verify that $\rho \approx 1$ for Euler's Method. **Upload these results to the** `Assignments -- Lab 2` **section of Blackboard**. For example, take a photo of the table and upload that. However, it would be better to upload a modified version of the `Euler.m` script that computes ρ for each n.

2.5.9 More MATLAB: formatted output

When you run the `Euler.m` script file, you'll see that the output is not very pretty. In particular, the data are not nicely tabulated. The script uses the `fprintf` function to display messages and values. Its basic syntax is:
`fprintf('Here is a message\n');`
where the `\n` indicates a new line.

Using `fprintf` to display the contents of a variable is a little more involved, and depends on *how* we want the data displayed. For example:

- To display an integer:
  `fprintf('Using n=%d steps\n', n);`
- To display an integer, padded to a maximum of 8 spaces:
  `fprintf('Using n=%8d steps\n', n);`
- To display a floating point number:
  `fprintf('y(n)=%f\n', Y(n+1));`
- To display in exponential notation:
  `fprintf('Error=%e\n', Error);`
- To display 3 decimal places:
  `fprintf('y(n)=%.3f\n', Y(n+1));`
  `fprintf('Error=%.3e\n', Error);`

Use these to improve the formatting of the output from your `Euler.m` script so that the output is nicely tabulated. To get more information on `fprintf`, type
`>> doc fprintf`
or ask one of the tutors.