Advanced Linear Algebra - Problem Set 2 (NM)

Exercise 2.1. Use the "column version" of matrix-matrix multiplication to show that the product of two lower-triangular matrices is lower-triangular.

We partition the identity matrix into columns as

$$I = (e_1|e_2|\cdots|e_n).$$

That is,

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad \cdots \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Exercise 2.2.

1. Show that Ae_j is the jth column on A.
2. Let v be the vector $v = (1, 1, \ldots, 1)$. What is Av?

The next exercise is from [1]. To answer it you need to know that a matrix A is

- involutory if $A^2 = I$,
- nilpotent if $A^k = 0$ for some integer $k > 0$
- a *projector* (also known as *idempotent*) if $A^2 = A$

Exercise 2.3.

1. Which is the only matrix that is both a projector and involutory?
2. Which is the only matrix that is both idempotent and nilpotent?
3. Prove that if A is a projector, then so too is $I - A$.
4. Prove that if A and B are both projectors, and $AB = BA$, them AB is also a projector.
5. Prove that A is involutory if and only if $(I - A)(I + A) = 0$.
6. Show that if A is involutory $B = \frac{1}{2}(I + A)$, then B is a projector.

Exercise 2.4. A matrix $L \in \mathbb{R}^{m\times m}$ is *lower triangular* if $l_{ij} = 0$ when $i < j$. If, in addition, $l_{ii} = 1$, then it is *unit lower triangular*. Show that the product of two unit lower triangular matrices is lower triangular.
Show that if we write the unit lower triangular matrix L as $L = I + N$, then N is nilpotent.
Show that

$$L^{-1} = I - N + N^2 - N^3 + \ldots.$$

Deduce that L^{-1} is itself a unit lower triangular matrix.

Exercise 3.1. Prove that $A \in \mathbb{C}^{m\times m}$ has full rank if and only if it maps no two distinct vectors to the same vector. That is, show that if x and y are vectors, with $x \neq y$, then

$$Ax \neq Ay \iff \operatorname{rank}(A) = m.$$

Exercise 3.2. Suppose you know the rank of the $m \times m$ matrices A and B. What, if anything, can you say about the rank of $C = AB$? Suppose A or B have full rank, what can you say about the rank of C?

Exercise 6.1. Show that if $A \in \mathbb{C}^{n\times n}$ is non-singular, then

$$(A^{-1})^\star = (A^\star)^{-1}.$$

Exercise 6.2. Find a definition of a vector space.

Exercise 6.3. A matrix $S \in \mathbb{C}^{m\times m}$ is "skew-hermitian" if $S^\star = -S$. Show that the eigenvalues of S are pure imaginary.

Exercise 7.1 ((Exercise 2.1 of [2])). Show that if a matrix is both triangular and unitary, then it is diagonal.

Hint. There are at least two approaches to this. One is to consider what the structure of the inverse and of the Hermetian transpose of an upper triangular matrix might look like (to simplify, consider just upper triangular matrices). The second is to assume the matrix is upper triangular, and then check what conditions its second column must satisfy in order to be orthogonal to its first.

Exercise 10.1. Define the function $\|\cdot\|_{A,1} : \mathbb{C}^m \to \mathbb{R}$ as $\|x\|_{A,1} = \|Ax\|_1$ where $\|\cdot\|_1$ is the usual 1-norm (a.k.a. "the taxicab norm") on $\mathbb{C}^m$, and $A \in \mathbb{C}^{m\times m}$ is non-singular. Is it true that $\|Ax\|_1$ is a norm on $\mathbb{C}^m$?

Exercise 10.2. Show that the following inequalities hold for all vectors $x \in \mathbb{C}^m$. If possible, give a nontrivial example for which equality holds.

1. $\|x\|_\infty \leq \|x\|_2$
2. $\|x\|_2 \leq \sqrt{m}\|x\|_\infty$
3. $\|x\|_2 \leq \|\sqrt{x\|_1\|x\|_\infty}$
4. $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$

Which, if any, of these inequalities extend to the matrix norms induced by these vector norms?

Exercise 11.1. Show that the function that maps $A \in \mathbb{R}^{m\times m}$ to $A \to |\det(A)|$ is *not* a matrix norm.

Exercise 11.2. Show that for any of subordinate matrix norm, $\|I\| = 1$. (That is, the norm of the identity matrix is 1). Is this also true of $\|\cdot\|_F$?

Exercise 11.3. Show that for any matrix $A \in \mathbb{R}^{m\times m}$,

1. $$\|A\|_\infty = \max_{i=1,\dots,m} \sum_{j=1}^m |a_{i,j}|.$$
2. $$\|A\|_1 = \max_{j=1,\dots,m} \sum_{i=1}^m |a_{i,j}|.$$

Exercise 11.4. Show that for any subordinate matrix norm, $\|A\| \geq |\lambda|$ for any eigenvalue λ of A.

Exercise 11.5. Consider the function $\|\cdot\| : \mathbb{R}^{m\times m} \to \mathbb{R}$ defined by

$$\|A\|_{\widetilde{\infty}} = \max_{i,j} |a_{ij}|.$$

Show that this is a norm. Show that it is *not* sub-multiplicative.

References

[1] Isle C.F. Ipsen, *Numerical matrix analysis: Linear systems and least squares.* Society for Industrial and Applied Mathematics, 2009.

[2] L.N. Trefethen and D. Bau III, *Numerical linear algebra* Society for Industrial and Applied Mathematics, 1997.