Advanced Linear Algebra - Problem Set 3 (NM)

Exercise 13.1. Prove that the product of two unitary matrices in $\mathbb{C}^{m\times m}$ is unitary.

Exercise 13.2. Write down the SVDs of the following matrices.

$$1.\ \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix} \qquad 2.\ \begin{pmatrix} 0 & 4 \\ -2 & 0 \end{pmatrix} \qquad 3.\ \begin{pmatrix} 4 & -2 \\ 0 & 0 \end{pmatrix}$$

Exercise 13.3. The matrices A and B in $\mathbb{C}^{m\times m}$are *unitarily equivalent* if there exists a untiary matrix, $Q \in \mathbb{C}^{m\times m}$ such that $A = QBQ^\star$.

1. Show that, if A and B are unitarily equivalent, then they have the same singular values.
2. Suppose that $A, B \in \mathbb{C}^{m\times m}$ have the same singular values. Must they be unitarily equivalent?

Exercise 20.1. We proved in class that the SVD of any matrix exists. Carefully read the details in the Lecture 4 of the textbook that demonstrate that, if the matrix is square and the singular values distinct, then the left and right singular vectors are uniquely determined up to complex sign.

Exercise 20.2. In class we proved that $\text{range}(A) = \text{span}(u_1, u_2, \ldots, u_r)$, where r is the number of nonzero singular values of A. Now prove that

$$\text{null}(A) = \text{span}(v_{r+1}, \ldots, v_n).$$

Exercise 22.1.

- Suppose that D is a diagonal matrix; i.e., $D = \text{diag}(d_{11}, d_{22}, \ldots, d_{mm})$. Show that each d_{ii} is an eigenvalue of D. What are the corresponding eigenvectors?
- Suppose that L is a lower triangular matrix. Show that each l_{ii} is an eigenvalue of L.

Exercise 23.1. Let A_v be the rank-v approximation to A

$$A_v = \sum_{j=1}^{r} \sigma_j u_j v_j^\star.$$

Prove that

$$\|A - A_v\|_F = \inf_{\text{rank}(X)\leq v} \|A - X\|_F = \sqrt{\sigma_{v+1}^2 + \sigma_{v+1}^2 + \cdots + \sigma_r^2}.$$

During this section of the course, we occasionally made use of arguments based on partitioning matrices. Use this idea to prove the following theorem.

Exercise 23.2. Suppose that onc can partition the matrix A as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}, \end{pmatrix}$$

where A and A_{11} are nonsingular. Let $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$. Show that

$$A^{-1} = \begin{pmatrix} A_11^{-1} + A_{11}^{-1}A_{12}S^{-1}A_{21}A_{11}^-1 & -A_{11}^{-1}A_{12}S^{-1} \\ -S^{-1}A_{21}A_{11}^{-1} & S^{-1}. \end{pmatrix}$$

Exercise 23.3. Recall the definition of a *lower triangular* and *unit lower triangular* matrices. Also, an *upper triangular* matrix is one whose transpose is lower triangular. Let L_1 and L_2 be unit lower triangular matrices. Let U_1 and U_2 be nonsingular upper triangular matrices. Suppose that $L_1U_1 = L_2U_2$. Show that $L_1 = L_2$ and $U_1 = U_2$.

Exercise 27.1. Show that

$$P = \begin{pmatrix} 0 & 0 \\ \alpha & 1 \end{pmatrix}$$

is a projector. For what α is it an orthogonal projector?

The following three questions are from Lecture 6 of Trefethen and Bau.

Exercise 27.2. Prove that, if P is an orthogonal projector, then $I - 2P$ is a unitary matrix.

Exercise 27.3. Write down the square matrix F such that

$$F \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} x_m \\ x_{n-1} \\ \vdots \\ x_1 \end{pmatrix}.$$

Let $E = (I + F)/2$. Is E a projector? Is it an orthogonal projector?

Exercise 27.4. Let $P \in \mathbb{C}^{m \times m}$be a non-zero projector. Prove that $\|P\|_2 \geq 1$. Prove that $\|P\|_2 = 1$ if, and only if, P is an orthogonal projector.

Exercise 30.1. Use that the square matrix A has a QR factorisation to prove that

$$|\det(A)| \leq \prod_{j=1}^{m} \|a_j\|_2,$$

where, as usual, a_j is column j of A.

Exercise 30.2. Let F be a Householder reflector. That is, for some vector v,

$$F = I - 2\frac{vv^\star}{v^\star v}.$$

Determine the eigenvalues, determinant, and singular values of F.

Exercise 31.1. Show that computing the matrix product $B = FA$, where F and A are both $m \times m$ matrices, using the "usual" algorithm, takes $\mathcal{O}(m^3)$ operations.

Let F be the Householder reflector

$$F = I - 2vv^\star,$$

where v is an m-vector such that $\|v\|_2 = 1$. Show that computing $B = FA$ is the same as $B = A - 2v(v^\star A)$. How many operations would this require?

Exercise 31.2. Use the existance of the Schur Form of the matrix $A \in \mathbb{C}^{m \times m}$ to prove that

$$\lim_{n \to \infty} \|A^n\| = 0 \iff \rho(A) < 1,$$

where $\rho(A)$ is the spectral radius of A.

Recall that the QR Algorithm is:

- Set $A^{(0)} = A$
- $k = 1, 2, 3, \ldots$
  - Compute $Q^{(k)}R^{(k)} = A$, the QR factorisation of A.
  - Set $A^{(k)} = Q^{(k)}R^{(k)}$

Exercise 34.1. Show that $A^{(k)}$ is similar A.

Exercise 34.2. Show that $A^k = \underline{Q}^{(k)}\underline{R}^{(k)}$, where

$$\underline{Q}^{(k)} = Q^{(1)}Q^{(2)} \cdots Q^{(k)}, \quad \text{and} \quad \underline{R}^{(k)} = R^{(k)}R^{(k-1)} \cdots R^{(1)}.$$