Advanced Linear Algebra - Algorithms

Niall Madden
School of Mathematics, Statistics and Applied Mathematics
National University of Ireland, Galway
Niall.Madden@NUIGalway.ie November 27, 2017These notes accompany the Algorithms section of Advanced Linear Algebra (MA500-1) taught at NUI Galway in Semester 1 2017/2018 by Niall Madden. For notes of the theory section, taught by Rachel Quinlan, please see http://www.maths.nuigalway.ie/~rquinlan/linearalgebra/.

1 Introduction (RQ)

RQ and NM introduced themselves, and introduced the module. RQ then discussed linear transformations and eigenvalues.

2 Matrix Multiplication (NM)

These section closely follows Lecture 1 of Trefethen and Bau's Numerical Linear Algebra. The first 5 lectures are freely available from the first author's home page. The summary below should be read in the context of those notes. I don't reproduce their text, just repeat the examples done in class.

Some of the following exercises are taken from Ilse Ipsen's Numerical Matrix Analysis. Full text is available online through the library portal.

2.1 Matrix vector multiplication

If A is an $m \times n$ matrix, and x and b are m-vectors, with given by $b = Ax$, then

$$b_i = \sum_{j=1}^{N} a_{ij} x_j.$$

Example 2.1. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

This gives

$$b = \begin{pmatrix} x_1 a + x_2 b \\ x_1 c + x_2 d \end{pmatrix} = x_1 \begin{pmatrix} a \\ c \end{pmatrix} + x_2 \begin{pmatrix} b \\ d \end{pmatrix}.$$

So b is a linear combination of the columns of A with coefficients from x.

2.2 Matrix-matrix multiplication

If A is a $m \times n$-matrix, C is a $n \times p$ matrix, and $B = AC$, then B is the $m \times p$ matrix given by

$$b_{ij} = \sum_{k=1}^{n} a_{ik} c_{kj}.$$

But in keeping with the ideas above, let us consider the formula for column j of B:

$$b_j = \sum_{k=1}^{n} c_{kj} a_k.$$

So column j of B is a linear combination of all the columns of A, with the coefficients taken from column j of C.

Other examples that are worth considering, include computing the inner and outer products of the vectors $(a, b, c)^T$ and $(d, e, f)^T$.

Example 2.2.

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$A^2 A^3 A^4$

Please also read Section 1.7 of [1] for more on four different "views" of matrix-matrix multiplication.

2.3 Exercises

Exercise 2.3. Use the "column version" of matrix-matrix multiplication to show that the product of two lower-triangular matrices is lower-triangular.

We partition the identity matrix into columns as

$$I = (e_1 | e_2 | \cdots | e_n).$$

That is,

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad \ldots \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Exercise 2.4.

1. Show that Ae_j is the jth column on A.
2. Let v be the vector $v = (1, 1, \ldots, 1)$. What is Av?

The next one is from [1]. To answer it you need to know that a matrix A is

- involutory if $A^2 = I$,
- nilpotent if $A^k = 0$ for some integer $k > 0$
- a *projector* (also known as *idempotent*) if $A^2 = A$

Exercise 2.5.

1. Which is the only matrix that is both a projector and involutory?
2. Which is the only matrix that is both idempotent and nilpotent?
3. Prove that if A is a projector, then so too is $I - A$.
4. Prove that if A and B are both projectors, and $AB = BA$, them AB is also a projector.

5. Prove that A is involutory if and only if $(I - A)(I + A) = 0$.
6. Show that if A is involutory $B = \frac{1}{2}(I + A)$, then B is a projector.

Exercise 2.6. A matrix $L \in \mathbb{R}^{m \times m}$ is *lower triangular* if $l_{ij} = 0$ when $i < j$. If, in addition, $l_{ii} = 1$, then it is *unit lower triangular.* Show that the product of two unit lower triangular matrices is lower triangular.

Show that if we write the unit lower triangular matrix L as $L = I + N$, then N is nilpotent.

Show that

$$L^{-1} = I - N + N^2 - N^3 + \dots.$$

Deduce that L^{-1} is itself a unit lower triangular matrix.

3 Range, nullspace, and rank (NM)

More from Lecture 1 of Trefethen and Bau.

3.1 Exercises

Exercise 3.1. Prove that $A \in \mathbb{C}^{m \times m}$ has full rank if and only if it maps no two distinct vectors to the same vector. That is, show that if x and y are vectors, with $x \neq y$, then

$$Ax \neq Ay \iff \operatorname{rank}(A) = m.$$

Exercise 3.2. Suppose you know the rank of the $m \times m$ matrices A and B. What, if anything, can you say about the rank of $C = AB$? Suppose A or B have full rank, what can you say about the rank of C?

4 The matrix of a linear transformation (RQ)

See http://www.maths.nuigalway.ie/~rquinlan/linearalgebra/lecture4.pdf

5 Some bases are better than others (RQ)

See http://www.maths.nuigalway.ie/~rquinlan/linearalgebra/lecture5.pdf

6 Orthogonal Vectors (NM)

6.1 Some bases are better than others

We recalled that, towards the end of Lecture 4, we observed that, if A is an $n \times n$ matrix with n linearly independent rows, then there exists a matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I.$$

We say that A is non-singular or invertible. We then noted that multiplying by A^{-1} (or, indeed, by A) can be interpreted as a change of basis operation.

6.2 Transpose

The transpose of the matrix $A \in \mathbb{R}^{m \times n}$ is $A^T \in \mathbb{R}$ where

$$(A)_{i,j} = (A^T)_{j,i}.$$

However, if $A \in \mathbb{C}^{m \times n}$ we need to define the related concept of the Hermetian Transpose, $A^\star$, for which

$$(A)_{i,j} = \overline{(A^T)_{j,i}}.$$

Here $\bar{z}$ denotes the *complex conjugate* of z. It is easy to show that

$$(AB)^\star = B^\star A^\star.$$

Exercise 6.1. Show that if $A \in \mathbb{C}^{n\times n}$ is non-singular, then

$$(A^{-1})^\star = (A^\star)^{-1}.$$

6.3 Vector spaces and inner product spaces

Exercise 6.2. Find a definition of a vector space.

An inner product space (of vectors over a field $\mathbb{F}$) is a vector space, V, equipped with the function $(\cdot,\cdot) : V \times V \to \mathbb{F}$, such that, if $x, y, z \in V$ and $a \in \mathbb{F}$ then

$$\begin{aligned}
(x, y) &= \overline{(x, y)} \\
(ax, y) &= a(x, y) \\
(x + y, z) &= (x, z) + (y, z) \\
(x, x) &\geq 0 \\
(x, x) &= 0 \text{ iff } x = 0.
\end{aligned}$$

There are many possible inner products, but the most important, for vectors with entries in $\mathbb{C}$ is the usual *dot product*:

$$(u, v) := u^\star v = \sum_{i=0}^{n} \bar{u}_i v_i.$$

For real-valued vectors it can be understood as the "angle" between the vectors in $\mathbb{R}$. That is, if α is the angle between two vectors, u and v, then,

$$\cos(\alpha) = \frac{(u, v)}{\sqrt{(u, u)}\sqrt{(v, v)}}.$$

The most important/interesting situation when $(u, v) = 0$, in which case we say that u and v are *orthogonal*.

6.4 Exercises

Exercise 6.3. A matrix $S \in \mathbb{C}^{m\times m}$ is "skew-hermitian" if $S^\star = -S$. Show that the eigenvalues of S are pure imaginary.

7 Unitary Matrices (NM, 18 Sep)

Definition 7.1. A matrix $Q \in \mathbb{C}^{m\times m}$ is **unitary** if its columns form an orthogonal, orthonormal set.

It follows that $Q^{-1} = Q^\star$.

7.1 Exercise(s)

Exercise 7.2 ((Exercise 2.1 of [2])). Show that if a matrix is both triangular and unitary, then it is diagonal.

Hint. There are at least two approaches to this. One is to consider what the structure of the inverse and of the Hermetian transpose of an upper triangular matrix might look like (to simplify, consider just upper triangular matrices). The second is to assume the matrix is upper triangular, and then check what conditions its second column must satisfy in order to be orthogonal to its first.

8 Symmetric Matrices I (RQ, 19 Sep)

See http://www.maths.nuigalway.ie/~rquinlan/linearalgebra/lecture8-9.pdf

9 Symmetric Matrices II (RQ, 20 Sep)

See http://www.maths.nuigalway.ie/~rquinlan/linearalgebra/lecture8-9.pdf

10 Vector Norms (NM, 25 Sep)

Definition 10.1 (Norm on $\mathbb{C}^m$)**.** A function $\|\cdot\|$ is called a ***norm*** on $\mathbb{C}^m$ if, for all vectors x and y in $\mathbb{C}^m$

1. $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$.
2. $\|\lambda x\| = |\lambda|\|x\|$ for any scalar $\lambda \in \mathbb{C}\sim$.
3. $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

The norm of a vector gives us some information about the "*size*" of the vector. But there are different ways of measuring the size: you could take the absolute value of the largest entry, you cold look at the "distance" for the origin, etc...

Definition 10.2 (p-norms.)**.** For any $p < 1$, define

$$\|x\|_p = \big(\sum_{i=1}^{m} |x_i|^p\big)^{1/p}.$$

Of particular importance are the norms

1. The 1-norm (or "taxi-cab norm"): $\|\vec{x}\|_1 = \sum_{i=1}^n |x_i|$.
2. The 2-norm (a.k.a. the *Euclidean norm*): $\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$. Note, if $x \in \mathbb{C}^m$, then $x^\star x = \|x\|_2^2$.
3. The ∞-norm (also known as the *max-norm*): $\|\vec{x}\|_\infty = \max_{i=1}^n |x_i|$.

It takes a little bit of effort to show that $\|\cdot\|_2$ satisfies the triangle inequality. First we need the Cauchy-Schwarz inequality.

Lemma 10.3 (Cauchy-Schwarz inequality)**.** *For all $x, y \in \mathbb{C}^m$,*

$$|x^T y| \leq \|x\|_2 \|y\|_2.$$

This can then be applied to show that

$$\|x + y\|_2 \leq \|x\|_2 + \|y\|_2.$$

It follows directly that $\|\cdot\|_2$ is a norm.

10.1 Exercises

Exercise 10.4. Define the function $\|\cdot\|_{A,1} : \mathbb{C}^m \to \mathbb{R}$ as $\|x\|_{A,1} = \|Ax\|_1$ where $\|\cdot\|_1$ is the usual 1-norm (a.k.a. "the taxicab norm") on $\mathbb{C}^m$, and $A \in \mathbb{C}^{m\times m}$ is non-singular. Is it true that $\|Ax\|_1$ is a norm on $\mathbb{C}^m$?

Exercise 10.5. Show that the following inequalities hold for all vectors $x \in \mathbb{C}^m$. If possible, give a nontrivial example for which equality holds.

1. $\|x\|_\infty \leq \|x\|_2$
2. $\|x\|_2 \leq \sqrt{m}\|x\|_\infty$
3. $\|x\|_2 \leq \sqrt{\|x\|_1\|x\|_\infty}$
4. $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$

Which, if any, of these inequalities extend to the matrix norms induced by these vector norms?

11 Matrix Norms (NM, 26 Sep)

Definition 11.1. A **matrix norm**...

11.1 Exercises

Exercise 11.2. Show that the function that maps $A \in \mathbb{R}^{m\times m}$ to $A \to |\det(A)|$ is *not* a matrix norm.

Exercise 11.3. Show that for any of subordinate matrix norm, $\|I\| = 1$. (That is, the norm of the identity matrix is 1). Is this also true of $\|\cdot\|_F$?

Exercise 11.4. Show that for any matrix $A \in \mathbb{R}^{m\times m}$,

1. $$\|A\|_\infty = \max_{i=1,\dots,m} \sum_{j=1}^{m} |a_{i,j}|.$$

2. $$\|A\|_1 = \max_{j=1,\dots,m} \sum_{i=1}^{m} |a_{i,j}|.$$

Exercise 11.5. Show that for any subordinate matrix norm, $\|A\| \geq |\lambda|$ for any eigenvalue λ of A.

Exercise 11.6. Consider the function $\|\cdot\| : \mathbb{R}^{m\times m} \to \mathbb{R}$ defined by

$$\|A\|_{\widetilde{\infty}} = \max_{i,j} |a_{ij}|.$$

Show that this is a norm. Show that it is *not* sub-multiplicative.

12 Diagonalizability of real symmetric matrices (RQ, 27 Sep)

See http://www.maths.nuigalway.ie/~rquinlan/linearalgebra/

13 Introduction to the Singular Value Decomposition (NM, 2 Oct)

13.1 Singular Values

The singular values $\{\sigma_1, \sigma_2, \dots, \sigma_m\}$ of a matrix A can be defined in several ways:

- the square roots of the eigenvalues of $B = A^\star A$,
- the lengths of the semiaxes of the hyperellipse that is the image of the unit circle under the linear transformation defined by A.
- The diagonal entries of the diagonal matrix Σ when we write A has a factorisation $A = U\Sigma V^\star$ where U and V are unitary matrices, and Σ is a diagonal matrix. (We'll be a little more precise below).

Please read Lecture 4 of Trefethen and Bau. T+B make the important observation in their introduction: *the image of the unit sphere under any $m \times n$ matrix is a hyperellipse.*

Suppose that $\{u_i\}_{i=1}^m$ is a set of orthonormal vectors, such that principal semiaxes of the hyperellipse are $\{\sigma_i u_i\}$. These scalars $\sigma_1, \sigma_2, \dots, \sigma_m$ are the *singular values of* A, and encapsulate many important properties of A.

13.2 Singular Value Decomposition

The rest of the exposition followed Lecture 4 of Trefethen and Bau pretty closely. In summary, the SVD of $A \in \mathbb{C}^{m\times n}$ is

$$AV = \Sigma U,$$

where

- $U = (u_1|u_2|u_3|\dots|u_m)$, the matrix that has as its columns the orthoronormal vectors in the direction of the principal semiaxes of the image of the unit sphere $S \in \mathbb{R}^n$.
- $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$ is the diagonal matrix containing the lengths of the principal semiaxes. We order them as
$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0.$$
- Then $V = (v_1|v_2|v_3|\dots|v_n)$ is the matrix whose columns are the preimages of the principal semiaxes, i.e., $Av_i = \sigma_i u_i$. It should be clear that the $\{v_i\}$ form an orthonormal set, so $V^{-1} = V^\star$.

13.3 Exercises

Exercise 13.1. Prove that the product of two unitary matrices in $\mathbb{C}^{m\times m}$ is unitary.

Exercise 13.2. Write down the SVDs of the following matrices.

1. $\begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$
2. $\begin{pmatrix} 0 & 4 \\ -2 & 0 \end{pmatrix}$
3. $\begin{pmatrix} 4 & -2 \\ 0 & 0 \end{pmatrix}$

Exercise 13.3. The matrices A and B in $\mathbb{C}^{m\times m}$are *unitarily equivalent* if there exists a untiary matrix, $Q \in \mathbb{C}^{m\times m}$ such that $A = QBQ^\star$.

1. Show that, if A and B are unitarily equivalent, then they have the same singular values.
2. Suppose that $A, B \in \mathbb{C}^{m\times m}$ have the same singular values. Must they be unitarily equivalent?

14 No class (3 Oct)

Workshop on supporting learning in mathematics.

15 Tutorial (RQ, 4 Oct)

16 Positive (semi)definite matrices (RQ, 9 Oct)

See http://www.maths.nuigalway.ie/~rquinlan/linearalgebra/lecture1617.pdf

17 Positive (semi)definite matrices (RQ, 10 Oct)

See http://www.maths.nuigalway.ie/~rquinlan/linearalgebra/lecture1617.pdf

18 Existence of the Singular Value Decomposition (NM, 11 Oct)

This class when rather slowly, indeed. It was supposed to culminate with a proof of the following.

Theorem 18.1 (The existence of the SVD)**.** *Every matrix has an SVD, and the singular values are uniquely determined. If the matrix is square and the singular values distinct, the left and right singular vectors are uniquely determined up to complex sign.*

However, the timing was very poor, and so we will have to return to this again

19 Ophelia (N/A, 16 Oct)

20 Yet more about the SVD (NM, 17 Oct)

We returned to the topic of the SVD, and we finally completed the proof of its existence. After that, we proved two short results.

Theorem 20.1. *The rank of A is the number of nonzero singular values.*

Theorem 20.2. $\text{range}(A) = \text{span}(u_1, u_2, \dots, u_r)$*, and* $\text{null}(A) = \text{span}(v_{r+1}, \dots, v_n)$*, where r is the number of nonzero singular values of A.*

20.1 Exercises

Exercise 20.3. We proved in class that the SVD of any matrix exists. Carefully read the details in the Lecture 4 of the textbook that demonstrate that, if the matrix is square and the singular values distinct, then the left and right singular vectors are uniquely determined up to complex sign.

Exercise 20.4. In class we proved that $\text{range}(A) = \text{span}(u_1, u_2, \dots, u_r)$, where r is the number of nonzero singular values of A. Now prove that

$$\text{null}(A) = \text{span}(v_{r+1}, \dots, v_n).$$

21 Stuff (RQ, 18 Oct)

22 Some theorems about the SVD (NM, 23 Oct)

In what follows we will assume each matrix $A \in \mathbb{C}^{m\times n}$ and that $p = \min(m, n)$.Recall the *Frobenius norm* of a matrix:

$$\|A\|_F := \Big(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\Big)^{1/2}.$$

Theorem 22.1.
$$\|A\|_2 = \sigma_1 \quad \text{and} \quad \|A\|_F = \left(\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_p^2\right)^{1/2}.$$

Theorem 22.2. *The singular values of A are the square roots of the eigenvalues of $A^\star A$.*

Theorem 22.3. *If A is hermitian, the singular values of A are the absolute values of the eigenvalues of A.*

Theorem 22.4. *If $A \in \mathbb{C}^{m\times m}$, then $|\det(A)| = \prod_{i=1}^m \sigma_i$.*

We concluded with noting that A can be written as the sum of rank 1 matrices, and idea we will return to in the next class.

22.1 Exercises

Exercise 22.5.

- Suppose that D is a diagonal matrix; i.e., $D = \operatorname{diag}(d_{11}, d_{22}, \dots, d_{mm})$. Show that each d_{ii} is an eigenvalue of D. What are the corresponding eigenvectors?
- Suppose that L is a lower triangular matrix. Show that each l_{ii} is an eigenvalue of L.

23 Low Rank Approximations (NM, 24 Oct)

First we proved the following result.

Theorem 23.1. *A is the sum of the r rank-one matrices*
$$A = \sum_{j=1}^{r} \sigma_j u_j v_j^\star.$$

Then we moved on to the following, highly important theorem.

Theorem 23.2. *Let A_V be the rank-v approximation to A*
$$A_v = \sum_{j=1}^{r} \sigma_j u_j v_j^\star.$$

Then
$$\|A - A_v\|_2 = \inf_{\operatorname{rank}(X)\le v} \|A - X\|_2 = \sigma_{v+1},$$
where if $v = p = \min(m,n)$, and we define $\sigma_{v+1} = 0$.

A similar result is true for the $\|\cdot\|_F$ norm, the proof of which is left to an exercise.

Theorem 23.3.
$$\|A - A_v\|_F = \inf_{\operatorname{rank}(X)\le v} \|A - X\|_F = \sqrt{\sigma_{v+1}^2 + \sigma_{v+1}^2 + \cdots + \sigma_r^2}.$$

All the above results emphasise the power the power of the SVD as an important theoretical and computational tool. We demonstrated this in class by using the SVD to compute "approximations" of certain images.

Computing the SVD is an important task, but not a trivial one. There are a few different approaches, but a key idea is the *QR-factorisation*.

23.1 Exercises

Exercise 23.4. Prove Theorem 23.3.

During this section of the course, we occasionally made use of arguments based on partitioning matrices. Use this idea to prove the following theorem.

Exercise 23.5. Suppose that onc can partition the matrix A as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}, \end{pmatrix}$$

where A and A_{11} are nonsingular. Let $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$. Show that

$$A^{-1} = \begin{pmatrix} A_11^{-1} + A_{11}^{-1}A_{12}S^{-1}A_{21}A_11^{-}1 & -A_{11}^{-1}A_{12}S^{-1} \\ -S^{-1}A_{21}A_{11}^{-1} & S^{-1}. \end{pmatrix}$$

Exercise 23.6. Recall the definition of a *lower triangular* and *unit lower triangular* matrices in Exercise 2.6. Also, an *upper triangular* matrix is one whose transpose is lower triangular. Let L_1 and L_2 be unit lower triangular matrices. Let U_1 and U_2 be nonsingular upper triangular matrices. Suppose that $L_1U_1 = L_2U_2$. Show that $L_1 = L_2$ and $U_1 = U_2$.

24 Positive matrices - the Perron-Frobenius Theorem. (RQ, 25 Oct)

See `http://www.maths.nuigalway.ie/~rquinlan/linearalgebra/lecture24.pdf`

25 Proof of the Perron-Frobenius Theorem. (RQ, 31 Oct)

See `http://www.maths.nuigalway.ie/~rquinlan/linearalgebra/lecture2526.pdf`

26 Projectors (NM, 1 Nov)

We started with the idea of a *projector* matrix: P where $P^2 = P$. After that, it did not go well......

27 Orthogonal Projectors (NM, 6 Nov)

This went much better! We covered the following topics.

1. The square matrix P is a projector if $P^2 = P$. And $v \in \text{range}(P) \iff Pv = v$.
2. $I - P$is also a projector.
3. We (finally) noted that $\text{null}(P) = \text{range}(I - P)$.
4. $\text{range}(P) \cap \text{null}(P) = \{0\}$.
5. So P separates $\mathbb{C}^m$ into two subspaces, $S_1 = \text{range}(P)$ and $S_2 = \text{null}(p)$..
6. If $S_1 \perp S_2$ we call P and *orthogonal projector*. We proved that a projector P is orthogonal if, and only if, $P = P^\star$.
7. If $\hat{Q}$ is a matrix with orthogonal, orthonormal columns (though not necessarily square, so not unitary), then $P = \hat{Q}\hat{Q}^\star$ is an orthogonal projector.
8. For any vector q, with $\|q\|_2 = 1$, we can define the projectors

   $$P_q = qq^\star \qquad \text{and} \qquad P_{\perp q} = I - qq^\star.$$

   More generally, for any vector

   $$P_a = \frac{aa^\star}{a^\star a}.$$

27.1 Exercises

Exercise 27.1. Show that

$$P = \begin{pmatrix} 0 & 0 \\ \alpha & 1 \end{pmatrix}$$

is a projector. For what α is it an orthogonal projector?

The following three questions are from Lecture 6 of Trefethen and Bau.

Exercise 27.2. Prove that, if P is an orthogonal projector, then $I - 2P$ is a unitary matrix.

Exercise 27.3. Write down the square matrix F such that

$$F \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} x_m \\ x_{n-1} \\ \vdots \\ x_1 \end{pmatrix}.$$

Let $E = (I + F)/2$. Is E a projector? Is it an orthogonal projector?

Exercise 27.4. Let $P \in \mathbb{C}^{m \times m}$be a non-zero projector. Prove that $\|P\|_2 \geq 1$. Prove that $\|P\|_2 = 1$ if, and only if, P is an orthogonal projector.

28 More on the proof of the Perron-Frobenius Theorem. (RQ, 7 Nov)

See http://www.maths.nuigalway.ie/~rquinlan/linearalgebra/lecture2526.pdf

29 The end of the proof of the Perron-Frobenius Theorem. (RQ, 8 Nov)

30 The QR factorisation (NM, 13 Nov)

30.1 Exercises

Exercise 30.1. Use that the square matrix A has a QR factorisation to prove that

$$|\det(A)| \leq \prod_{j=1}^{m} \|a_j\|_2,$$

where, as usual, a_j is column j of A.

Exercise 30.2. Let F be a Householder reflector. That is, for some vector v,

$$F = I - 2\frac{vv^\star}{v^\star v}.$$

Determine the eigenvalues, determinant, and singular values of F.

31 The Schur Form (NM, 14 Nov)

31.1 Exercises

Exercise 31.1. Show that computing the matrix product $B = FA$, where F and A are both $m \times m$ matrices, using the "usual" algorithm, takes $\mathcal{O}(m^3)$ operations.

Let F be the Householder reflector

$$F = I - 2vv^\star,$$

where v is an m-vector such that $\|v\|_2 = 1$. Show that computing $B = FA$ is the same as $B = A - 2v(v^\star A)$. How many operations would this require?

Exercise 31.2. Use the existance of the Schur Form of the matrix $A \in \mathbb{C}^{m\times m}$ to prove that

$$\lim_{n\to\infty} \|A^n\| = 0 \iff \rho(A) < 1,$$

where $\rho(A)$ is the spectral radius of A.

32 Applications of Perron-Frobenius (I). (RQ, 15 Nov)

33 Applications of Perron-Frobenius (II). (RQ, 20 Nov)

34 Hessenberg Form and the QR Algorithm. (NM, 21 Nov)

34.1 Exercise

Recall that the QR Algorithm is:

- Set $A^{(0)} = A$
- $k = 1, 2, 3, \dots$
    - Compute $Q^{(k)}R^{(k)} = A$, the QR factorisation of A.
    - Set $A^{(k)} = Q^{(k)}R^{(k)}$

Exercise 34.1. Show that $A^{(k)}$ is similar A.

Exercise 34.2. Show that $A^k = \underline{Q}^{(k)}\underline{R}^{(k)}$, where

$$\underline{Q}^{(k)} = Q^{(1)}Q^{(2)}\cdots Q^{(k)},$$

and

$$\underline{R}^{(k)} = R^{(k)}R^{(k-1)}\cdots R^{(1)}.$$

35 Review (RQ + NM, 22 Nov)

References

[1] Ilse C.F. Ipsen, *Numerical matrix analysis: Linear systems and least squares.* Society for Industrial and Applied Mathematics, 2009.

[2] L.N. Trefethen and D. Bau III, *Numerical linear algebra* Society for Industrial and Applied Mathematics, 1997.