Predicting flow and river height after a junction

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Stokes Modelling Workshop, 2017
National University of Ireland, Galway

June, 2017

The Problem

The following is the problem as posed by Seshu Tirupathi of IBM Research.

Shallow water equations describing unsteady open channel flows are typically used to model river flows. Conservative form of the equations are difficult to numerically formulate and apply in real test cases. The conservative form of the equations are given by

$$\begin{aligned}
\frac{\partial(\rho\eta)}{\partial t}+\frac{\partial(\rho\eta u)}{\partial x}+\frac{\partial(\rho\eta v)}{\partial y}&=0,\\
\frac{\partial(\rho\eta u)}{\partial t}+\frac{\partial}{\partial x}\big(\rho\eta u^2+\frac{1}{2}\rho g\eta^2\big)+\frac{\partial(\rho\eta uv)}{\partial y}&=0,\\
\frac{\partial(\rho\eta v)}{\partial t}+\frac{\partial(\rho\eta uv)}{\partial x}+\frac{\partial}{\partial y}\big(\rho\eta v^2+\frac{1}{2}\rho g\eta^2\big)&=0,
\end{aligned}$$

where η is the total fluid column height (instantaneous fluid depth as a function of x, y and t), and the 2D vector (u,v) is the fluid's horizontal flow velocity, averaged across the vertical column. Further, g is acceleration due to gravity and ρ is the fluid density. On the other hand, the non conservative form of the equations are easier to formulate numerically. These are written as follows:

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}\Big((H+h)u\Big)+\frac{\partial}{\partial y}\Big((H+h)v\Big)=0,$$

$$\begin{aligned}
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-fv&=-g\frac{\partial h}{\partial x}-bu+\nu\Big(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\Big),\\
\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+fu&=-g\frac{\partial h}{\partial y}-bv+\nu\Big(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\Big),
\end{aligned}$$

where u is the velocity in the x direction, v is the velocity in the y direction, h is the height deviation of the horizontal pressure surface from its mean height H: $\eta = H+h$, H is the mean height of the horizontal pressure surface, g is the acceleration due to gravity, f is the Coriolis coefficient associated with the Coriolis force, on Earth equal to $2\Omega sin(\phi)$, where Ω is the angular rotation rate of the Earth ($\pi/12$ radians/hour), ϕ is the latitude, b is the viscous drag coefficient, and ν is the kinematic viscosity.

There have been various attempts to handle both these formulations. It is important to note that the one dimensional model decreases the computational cost significantly to model river flows to a reasonable accuracy as compared to a more comprehensive 2D resolution. The problem becomes tricky when there is confluence of river segments/channels. Work has been done to handle merging of flows from different channels into a single channel by using a 2D formulation at the junction. For this workshop, can you comment on some or all of the following aspects (assuming continuous flow):

1. In general, are there any disadvantages of using the non-conservative form of the equations even if the flow is continuous?

2. For flow at a junction, is it possible to come up with internal boundary conditions/constraints without the necessity to solve the 2D shallow water equations? How do the boundary conditions compare with a 1D-2D coupling at the junction? See Figure 1.

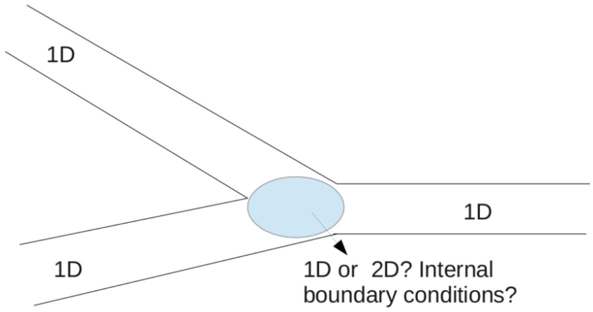

Figure 1: Flow at a junction

3. How do the internal boundary conditions/constraints compare when the flow merges at a cross section? See Figure 2.

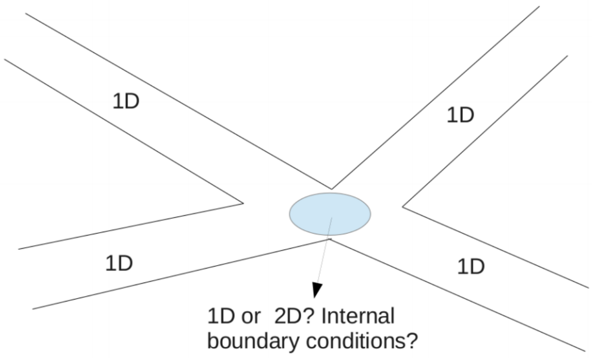

Figure 2: Flow at a cross section

4. Can you come up with a rigorous analysis for questions 1 to 3 assuming rectangular/trapezoidal/parabolic cross sections?

5. What other novel methods can you come up with to handle merging flows while keeping the formulation limited to 1D?

Approach

1D shallow water rectangular cross section

We first consider the 1D shallow water equations for a rectangular cross section perpendicular to the flow. Let the square cross section vary with x, the direction along the river flow, and time t according to

$$A(x,t) = w(x)h(x,t), \tag{1}$$

where $w(x)$ is the channel width, which is known, and $h(x,t)$ is the channel height which is unknown.

The 1D shallow water equations then become

$$w(x)\frac{\partial h(x,t)}{\partial t} + \frac{\partial (w(x)h(x,t))}{\partial x} = 0, \tag{2}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial (h+b)}{\partial x} = -\frac{(2h+w)}{hw}\frac{\tau}{\rho}, \tag{3}$$

where $u(x,t)$ is the flow velocity, $b(x)$ is the height of the river bed and $\tau(x,t)$ is the wall shear stress along the river basin. For an open channel, the Darcy–Weisbach friction becomes

$$\tau(x,t) = \frac{f_D}{8}\rho u(x,t)^2, \tag{4}$$

where f_D is the Darcy friction factor and $u(x,t)$ is the mean flow velocity along the basin.

What do we get for constant width w and the river bed height $b(x) = \beta x$, where β is constant? Substituting 4 into 3, and assuming a steady flow, so that u is constant in both x and t, and constant height h gives

$$u^2 = \frac{8hw}{(2h+w)}\frac{g}{f_D} \mid \beta \mid . \tag{5}$$

The sign of β is determined by the direction of the slope of the channel. The larger the incline and gravity, the faster the flow u. Finally, the larger the volume ratio to perimeter, and smaller the friction, the faster the flow. Note: $f_D \to 0$ doesn't make sense as we assume the flow velocity is constant and without friction this would not be possible (gravity would accelerate the flow).

How far can we get using $b(x) = \beta x$ and $w(x) = \omega x + w_0$? With these constraints, the 1D equations are as follows:

$$(\omega x + w_0)\frac{\partial h(x,t)}{\partial t} + \omega h(x,t) + (\omega x + \omega_0)\frac{\partial h(x,t)}{\partial x} = 0, \tag{6}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g(\frac{\partial h}{\partial x} + \beta) = -\frac{(2h+w)}{hw}\frac{f_D}{8}u(x,t)^2. \tag{7}$$

A 3D fluid

The correct way to do this is to solve the full 3D fluid equations for a constant uniform flow. The 1D shallow water equations are the result of averaging this solution. However, to pose balance of momentum in 3D, we would need to revert back to the true 3D solution of a uniform flow. This is quite straightforward, but for simplicity we will use just the averaged 1D shallow water solution.

To apply the conservation of momentum to surfaces we need the full stress tensor:

$$\sum_i \frac{\partial u_i}{\partial x_i} = 0 \qquad \text{(incompressibility)} \tag{8}$$

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad \text{(stress tensor)} \tag{9}$$

where p is the pressure.

Conservation of mass over a control volume Ω becomes

$$\frac{d}{dt}\int_\Omega \rho \mathrm{d}u = -\int_{\partial\Omega} (\rho\vec{u}).\vec{n}\mathrm{d}A.$$

If we assume no source of mass in Ω, then the left-hand side of the above is zero. As the flows are 1D, we get

$$\vec{u}_1 \cdot \vec{n}_1 = \pm u_1, \quad \vec{u}_2 \cdot \vec{n}_2 = \pm u_2 \quad \textit{and} \quad \vec{u}_3 \cdot \vec{n}_3 = \mp u_3 \sin\gamma, \tag{10}$$

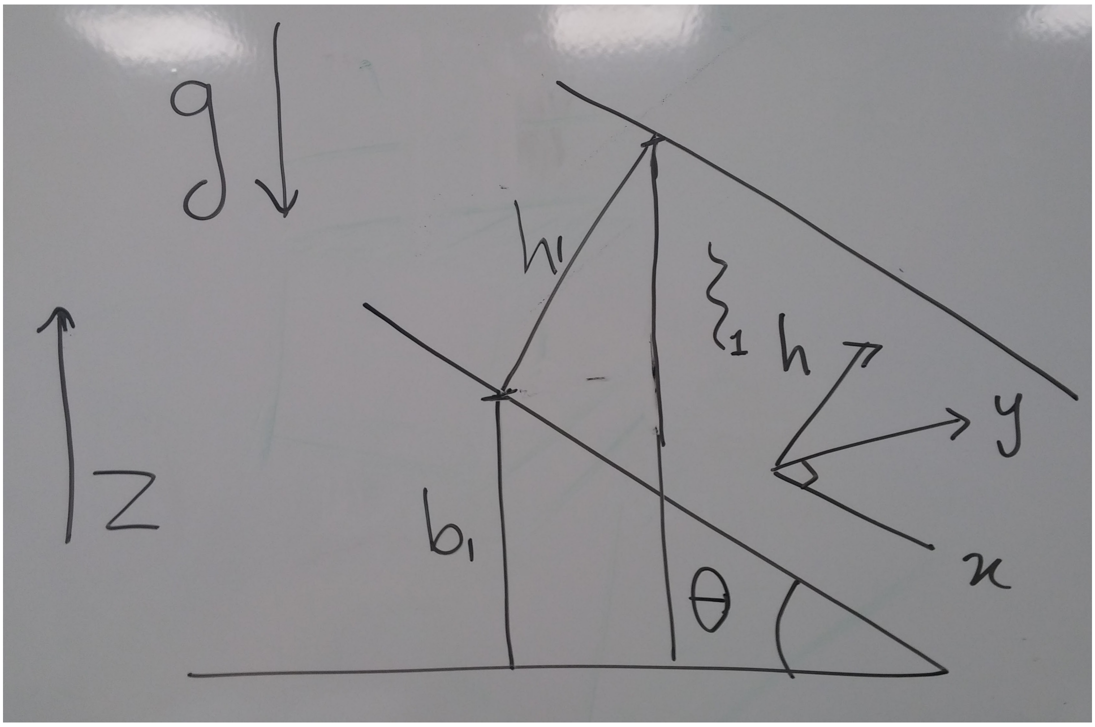

Figure 3:

which combined in the above give

$$u_1 A_1 + u_2 A_2 = u_3 A_3 \sin\gamma. \tag{11}$$

Conservation of momentum:

$$\frac{d}{dt}\int_\Omega \rho\vec{u}dv = -\int_{\partial\Omega}(\rho\vec{u})\vec{u}\cdot\vec{n}da + \int_\Omega \rho\vec{g}dv + \int_{\partial\Omega}\vec{\sigma}\cdot\vec{n}da, \tag{12}$$

where the body force, in the coordinate system (x, y, h) shown in Figure 3, is given by $\vec{g} = g(\sin\theta, 0, -\cos\theta)$.

Using balance of momentum and incompressibility, we can reduce the above to the local form,

$$\rho\frac{\partial\vec{u}}{\partial t} + \rho(\vec{u}\cdot\nabla)\vec{u} = -\nabla p + \rho\mu\nabla^2\vec{u} + \rho\vec{g}. \tag{13}$$

Considering no source of momentum, except gravity, then the lefthand side of the above is zero. All other terms are known except the shear force on the bottom surface of $\partial\Omega$. If the full equations of uniform flow were solved, we could estimate the flow on the bottom. We use a no-slip boundary condition on the river bed.

We separate the surface $\partial\Omega = \partial\Omega_1 + \partial\Omega_2 + \partial\Omega_3 + \mathcal{B} + \mathcal{S}$, where $\mathcal{B}$ is the river bed, $\mathcal{S}$ is the river surface, and each Ω_i is a face of the prism in Figure 4.

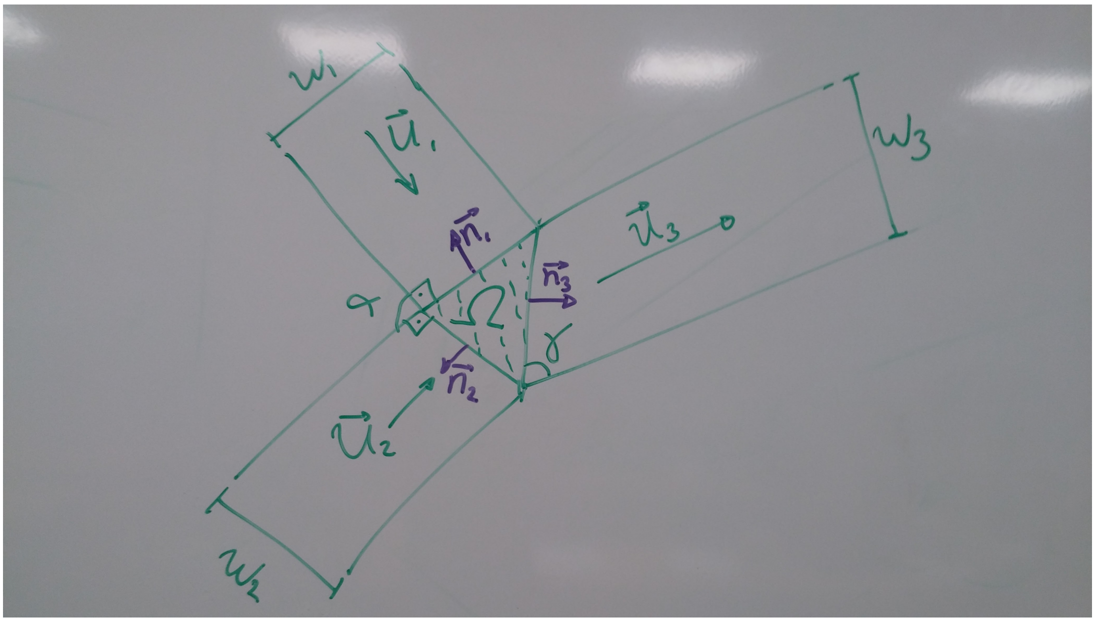

Figure 4:

The relevant boundary conditions are

$$\vec{n}_b(x,y) = (0,0,-1)\,, \quad \text{(normal vector to river base)}, \tag{14}$$
$$\vec{n}_s(x,y) = (0,0,1)\,, \quad \text{(normal vector to river surface)}, \tag{15}$$
$$\vec{u}(x,y,0)\cdot\vec{n}_b = 0, \quad \text{(no normal flow through base)}, \tag{16}$$
$$\vec{n}_b(x,y)\cdot\vec{\sigma}(x,y,0) = -p(x,y,0)\vec{I}, \quad \text{(no-slip on river bed)}, \tag{17}$$
$$\vec{u}(x,y,0)\cdot\vec{n}_s = -\frac{\partial\zeta(x,y,t)}{\partial t}, \quad \text{(no normal flow through surface)}, \tag{18}$$
$$\vec{n}_s(x,y)\cdot\vec{\sigma}(x,y,h) = 0, \quad \text{(no surface stress)}. \tag{19}$$

Taking the first integral on the right from conservation of momentum (12), we get

$$\int_{\partial\Omega_1+\partial\Omega_2+\partial\Omega_3}(\rho\vec{u})\vec{u}\cdot\vec{n}da = \pm\rho\vec{u}_1A_1u_1 \pm \rho\vec{u}_2A_2u_2 \mp \rho\vec{u}_3A_3u_3\sin\gamma.$$

Assuming no sources leads to $\partial\zeta/\partial t = 0$, the no normal flux boundary conditions lead to

$$\int_{\mathcal{B}+\mathcal{S}}(\rho\vec{u})\vec{u}\cdot\vec{n}da = 0,$$

for the river bed and surface.

To calculate the volume integrals and estimate pressure on the river bed, we will assume an average height in Ω to be

$$\bar{h} = \frac{1}{3}(h_1+h_2+h_3). \tag{20}$$

This gives

$$\int_\Omega \rho\vec{g}dv = \vec{g}\frac{\rho\bar{h}}{2}w_1w_2\sin\alpha.$$

For the traction on the surface $\partial\Omega_i$, the stress tensor simplifies to $\vec{\sigma} = -p\vec{I}$, because the flow is uniform in each channel, at least up to the surface $\partial\Omega_i$. From scaling arguments, and balance of momentum (13) in the z-direction, we get

$$p = \rho g(\zeta_i - z) = \rho g \frac{h_i - h}{\cos\theta}, \tag{21}$$

where we used $\cos\theta(z-b) = h$ together with $\zeta_i - b_i = h_i/\cos\theta$.

With the above we find that

$$\int_{\partial\Omega_1} \vec{\sigma}\cdot\vec{n}da = -\int_0^{h_1}\int_0^{w_1} \rho g \frac{h_1 - h}{\cos\theta}\vec{n}_1 dy dh = -w_1 \frac{\rho g h_1^2}{2\cos\theta}\vec{n}_1, \tag{22}$$

with an analogous result for $\partial\Omega_2$. Similarly,

$$\int_{\partial\Omega_3} \vec{\sigma}\cdot\vec{n}da = -\frac{w_3}{\sin\gamma}\frac{\rho g h_3^2}{2\cos\theta}\vec{n}_3, \tag{23}$$

where the $\sin\gamma$ appears because the surface $\partial\Omega_3$ is not perpendicular to $\vec{u}_3$.

The traction integral over the surface $\mathcal{S}$ is zero because of zero traction on surface. For the river bed, the stress tensor is

$$\vec{\sigma} = -p\vec{I} = -\frac{\rho g \bar{h}}{\cos\theta}\vec{I},$$

which leads to

$$\int_{\partial\mathcal{B}} \vec{\sigma}\cdot\vec{n}da = -\frac{\rho g\bar{h}}{\cos\theta}\vec{n}_b\frac{w_1 w_2}{2}\sin\alpha. \tag{24}$$

Summing over all surfaces gives

$$\int_{\partial\Omega} \vec{\sigma}\cdot\vec{n}da = -w_1\frac{\rho g h_1^2}{2\cos\theta}\vec{n}_1 - w_2\frac{\rho g h_2^2}{2\cos\theta}\vec{n}_2 - \frac{w_3}{\sin\gamma}\frac{\rho g h_3^2}{2\cos\theta}\vec{n}_3 \\ - \frac{\rho g\bar{h}}{2\cos\theta}\vec{n}_b w_1 w_2 \sin\alpha. \tag{25}$$

Combining all these integrals into conservation of momentum (12) leads to

$$\pm\vec{u}_1 A_1 u_1 \pm \vec{u}_2 A_2 u_2 \mp \vec{u}_3 A_3 u_3 \sin\gamma = \vec{g}\frac{\bar{h}}{2}w_1 w_2\sin\alpha - \frac{g\bar{h}}{2\cos\theta}\vec{n}_b w_1 w_2 \sin\alpha \\ - w_1\frac{g h_1^2}{2\cos\theta}\vec{n}_1 - w_2\frac{g h_2^2}{2\cos\theta}\vec{n}_2 - \frac{w_3}{\sin\gamma}\frac{g h_3^2}{2\cos\theta}\vec{n}_3. \tag{26}$$

To relate the vectors, angles, and widths we use

$$\frac{w_1}{\sin\tau_1} = \frac{w_2}{\sin\tau_2} = \frac{w_3}{\sin\gamma\sin\alpha},$$
$$\vec{n}_1\cdot\vec{n}_2 = \cos\alpha, \quad \vec{n}_1\cdot\vec{n}_3 = -\cos\tau_2, \quad \vec{n}_2\cdot\vec{n}_3 = -\cos\tau_1,$$
$$\vec{u}_3\cdot\vec{n}_1 = u_3\sin(\tau_2-\gamma), \quad \vec{u}_3\cdot\vec{n}_2 = -u_3\sin(\tau_1+\gamma),$$

where τ_1 is the angle opposite w_1, τ_2 is the angle opposite w_2, and $\sin\gamma\sin\alpha$ equals the *sin* of the angle opposite w_3. We remove gravity, which balances with friction and pressure:

$$\pm u_1\vec{n}_1 A_1 u_1 \pm u_2\vec{n}_2 A_2 u_2 \mp \vec{u}_3 A_3 u_3 \sin\gamma = \\ - w_1\frac{g h_1^2}{2\cos\theta}\vec{n}_1 - w_2\frac{g h_2^2}{2\cos\theta}\vec{n}_2 - \frac{w_3}{\sin\gamma}\frac{g h_3^2}{2\cos\theta}\vec{n}_3. \tag{27}$$

Taking the dot product of the above with $\vec{n}_1$, $\vec{n}_2$, $\vec{n}_3$, and then $\vec{u}_3$, before substituting $A_1 = h_1 w_1$, $A_2 = h_2 w_2$, $A_3 = w_3 h_3 / \sin\gamma$ we get the following equations:

$$\begin{aligned}
&\pm u_1^2 h_1 w_1 \pm u_2^2 \cos\alpha h_2 w_2 \mp u_3^2 \sin(\tau_2 - \gamma) w_3 h_3 = \\
&\qquad - w_1 \frac{g h_1^2}{2\cos\theta} - w_2 \frac{g h_2^2}{2\cos\theta} \cos\alpha + \frac{w_3}{\sin\gamma} \frac{g h_3^2}{2\cos\theta} \cos\tau_2, \\
&\pm u_1^2 \cos\alpha h_1 w_1 \pm u_2^2 h_2 w_2 \pm u_3^2 \sin(\tau_1 + \gamma) w_3 h_3 = \\
&\qquad - w_1 \frac{g h_1^2}{2\cos\theta} \cos\alpha - w_2 \frac{g h_2^2}{2\cos\theta} + \frac{w_3}{\sin\gamma} \frac{g h_3^2}{2\cos\theta} \cos\tau_1, \\
&\mp u_1^2 \cos\tau_2 h_1 w_1 \mp u_2^2 \cos\tau_1 h_2 w_2 \mp u_3^2 \sin\gamma w_3 h_3 = \\
&\qquad - w_1 \frac{g h_1^2}{2\cos\theta} \cos\tau_2 + w_2 \frac{g h_2^2}{2\cos\theta} \cos\tau_1 - \frac{w_3}{\sin\gamma} \frac{g h_3^2}{2\cos\theta}, \\
&\pm u_1^2 \sin(\tau_2 - \gamma) h_1 w_1 \mp u_2^2 \sin(\tau_1 + \gamma) h_2 w_2 \mp u_3 w_3 h_3 = \\
&\qquad - w_1 \frac{g h_1^2}{2\cos\theta} \sin(\tau_2 - \gamma) + w_2 \frac{g h_2^2}{2\cos\theta} \sin(\tau_1 + \gamma) - w_3 \frac{g h_3^2}{2\cos\theta},
\end{aligned} \tag{28}$$

$$u_1 w_1 h_1 + u_2 w_2 h_2 = u_3 w_3 h_3. \tag{29}$$

Results

Equations (28) and (29) are solved for u_3 and h_3, across varying angles between the merging rivers, using Julia. This resulted in the output shown in Figure 5.

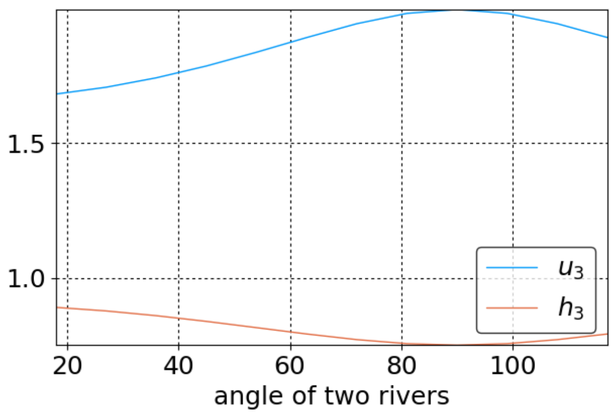

Figure 5: Changing the angle between the merging rivers

These equations are input into Julia again, this time solving for u_3 and h_3 across varying widths of the merged river. The resulting output is shown in Figure 6.

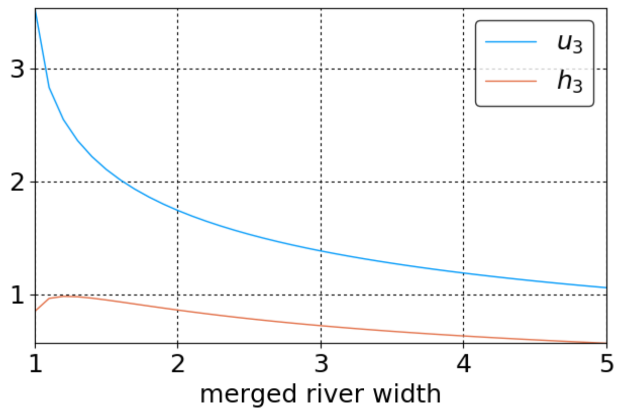

Figure 6: Changing the width of the merged river

Conclusion

A model of the flow at a river junction can be devised to predict the height and velocity at specific locations using the 1D shallow water equations. Using the flow into the merging region, we rely on the conservation of mass and momentum to predict the flow out of this region.

It later became apparent that we omitted enforcing the steady flow assumption 5 in our results. Combining the steady flow assumption with the conservation of mass would provide a more accurate prediction of the flow and river height out of the merging region.

Following an analysis of the resulting model, the method can then be extended, using simple geometry, to river junctions where the merging rivers do not connect to the merging region at right angles to their flow. The model can also be extended to deal with more complicated junctions by adding each additional river incrementally.