Theory & Computation of Singularly Perturbed Differential Equations

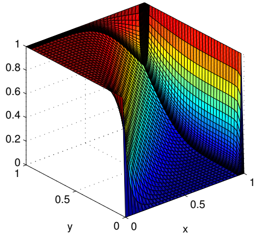
IIT (BHU) Varanasi, Dec 2017

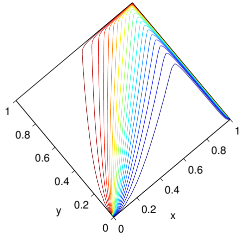
https://skumarmath.wordpress.com/gian-17/singular-perturbation-problems/

Niall Madden, NUI Galway

§1 Introduction to Singularly Perturbed Differential Equations

Version 02.12.17





Handout version.

Outline

Monday, 4 December		
09:30 – 10.30	Registration and Inauguration	
10:45 – 11.45	1. Introduction to singularly perturbed problems	NM
12:00 – 13:00	2. Numerical methods and uniform convergence	NM
14:30 – 15:30	Tutorial (Convection diffusion problems)	NM
15:30 – 16:30	Lab 1 (Simple FEMs in MATLAB)	NM

Tuesday, 5 December		
09:30 – 10:30	3. Finite difference methods and their analyses	NM
10:45 – 11:45	4. Coupled systems of SPPDEs	NM
14:00 – 16:00	Lab 2 (Fitted mesh methods for ODEs)	NM

Thursday, 7 December		
09:00 – 10:00	8. Singularly perturbed elliptic PDEs	NM
10:15 – 11:15	9. Finite Elements in two and three dimensions	NM
01:15 – 15:15	Lab 4 (Singularly perturbed PDEs)	NM

Friday, 8 December		
09:00 – 10:00	10. Preconditioning for SPPs	NM

§1 Introduction to singularly perturbed problems

(50 minutes)

When is a perturbation singular?

A regular perturbation

Consider the following example, taken from [O'Malley, 1997]:

$$x^2 + \varepsilon x - 1 = 0. \tag{1}$$

Here ε is the *perturbation parameter*. It is real and positive. In cases of interest it is small.

The solutions to (1) are

$$x = \frac{-\varepsilon \pm \sqrt{\varepsilon^2 + 4}}{2}. \tag{2}$$

If we let $\varepsilon \to 0$ in (1), the resulting problem has two solutions: $x = \pm 1$.

If we let $\varepsilon \to 0$ in (2), we again get $x = \pm 1$.

This is a regular perturbation

When is a perturbation singular?

A singular perturbation

Now consider a similar problem, but with the perturbation parameter multiplying the second-order term:

$$\varepsilon x^2 + x - 1 = 0. \tag{3}$$

The solutions to this problem are

$$x = \frac{-1 \pm \sqrt{1 + 4\varepsilon}}{2\varepsilon}. \tag{4}$$

If we set $\varepsilon = 0$ in (3), the resulting problem has a single solution: $x = 1$.

But if we let $\varepsilon \to 0$ in (4), the solutions tend to 1 and $-\infty$.

This is a singular perturbation

(*A similar explanation is given by Peter D. Miller (Michigan) in "Perturbation theory and asymptotics", §IV.5 of The Princeton Companion to Applied Mathematics.*)

Singularly Perturbed DEs

Compare the following two differential equations:

$$-(1+\varepsilon)u''(x)+u(x)=f(x) \quad \text{on } (0,1), \quad \text{with } u(0)=u(1)=0. \tag{5}$$

and

$$-\varepsilon u''(x)+u(x)=f(x) \quad \text{on } (0,1), \quad \text{with } u(0)=u(1)=0. \tag{6}$$

If we set $\varepsilon=0$ in (5), nothing remarkable happens: we still have a well-posed ODE.

But if we set $\varepsilon=0$ in (6), the problem is not well-posed, since, unless $f(0)=f(1)=0$, we cannot satisfy $u=f$ and the boundary conditions.

Question: What happens to (6) as $\varepsilon \to 0$?
Answer: Solutions develop "layers".

Linß provides a formal definition.

Singular functions [Linß, 2010, p2]

Let B be a function space with norm $\|\cdot\|_B$. Let $D \subset \mathbb{R}^d$ be a parameter domain. The continuous function $u : D \to B, \varepsilon \to u(\varepsilon)$ is said to be *regular* for $\varepsilon \to \varepsilon^\star \in \partial D$ if there exists a function $u^\star \in B$ such that:

$$\lim_{\varepsilon \to \varepsilon^\star} \|u_\varepsilon - u^\star\|_B = 0.$$

Otherwise u_ε is said to be **singular for $\varepsilon \to \varepsilon^\star$.**

Singularly perturbed [Linß, 2010, p3]

Let (P_ε) be a problem with solution $u(\varepsilon) \in B$ for all $\varepsilon \in D$. We say (P_ε) is **Singularly perturbed** for $\varepsilon \to \varepsilon^\star \in \partial D$ in the norm $\|\cdot\|_B$ if u is singular for $\varepsilon \to \varepsilon^\star$.

(Although we won't dwell on the point right now, it *is* important to note that the concept is norm-dependent).

The following, slightly less formal, characterisation is provided by Roos et al.

Singularly perturbed problems [Roos et al., 2008, p2]

[**Singularly perturbed problems**] are differential equations (ordinary or partial) that depend on a small positive parameter, ε, and whose solutions (or their derivatives) approach a discontinuous limit as ε approaches zero. Such problems are said to be singularly perturbed, where we regard ε as a **perturbation parameter**.

Neither of the above definitions/characterisations discuss what happens as $\varepsilon \to 0$. In this section, we shall see that, typically, solutions possess *layers*: regions where the solution and/or its derivatives change rapidly.

The interest in SPPs stems from the fact the solutions have **layers**, and from related challenges:

- SPPs are used to model phenomena that feature interior and/or boundary layers, a term introduced by Ludwig Prandtl[1]
- Standard numerical schemes often compute poor approximations to solutions to SPDEs; sometimes they are terrible.
- Standard mathematical techniques can fail to give useful results. ("*We critically review the available error analysis in computational fluid dynamics (CFD) and come to the conclusion that the existing error estimates are meaningless in most cases of interest*" [2]).

The remainder of this section is devoted to SPDEs and their layers. Their numerical solution will require more coffee.

[1] Prandtl. *On the motion of a fluid with very small viscosity*, Third World Congress of Mathematicians, August 1903.

[2] Johnson et al. *Numerics and Hydrodynamic Stability: Toward Error Control in Computational Fluid Dynamics*, SINUM 1995. [Johnson et al., 1995]

Our first example is a simple *reaction-diffusion* equation.

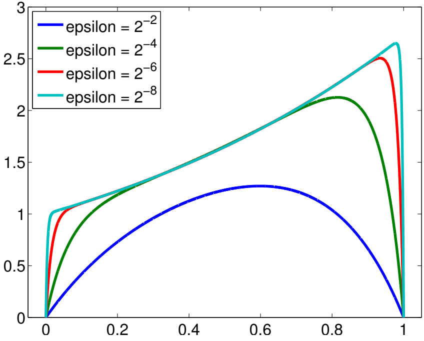
A singularly perturbed differential equation

$$-\varepsilon^2 u''(x) + u(x) = e^x \quad \text{on } (0,1), \quad \text{with } u(0) = u(1) = 0. \qquad (7)$$

The solutions to this equation look like

$$\underbrace{c_0 e^{-x/\varepsilon}}_{\text{left layer}} + \underbrace{c_1 e^{-(1-x)/\varepsilon}}_{\text{right layer}} + \underbrace{(e^x)/(1-\varepsilon^2)}_{\text{regular part}}$$

The first two terms are "layer terms", that decay rapidly away from the boundaries.
The third term is close to the solution of the "reduced" problem, obtained by setting $\varepsilon = 0$, and neglecting the boundary conditions.



The Bestiary

Here follows an incomplete list of SPDEs, with graphs of their solutions, and some notes about what makes them interesting.

This will include

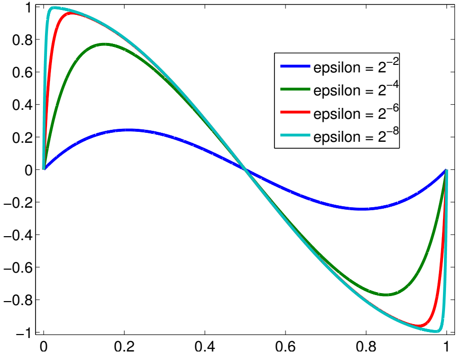
1. reaction-diffusion ODEs;
2. convection-diffusion problems;
3. coupled systems;
4. time-dependent (parabolic) problems;
5. all the above, but in two dimensions.

. . . .

An aside: [Collins English Dictionary] *A* ***bestiary*** *is a moralising medieval collection of descriptions (and often illustrations) of real and mythical animals.*

Example (A reaction-diffusion equation)

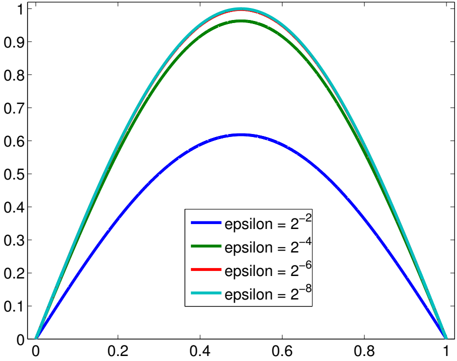
$$-\varepsilon^2 u''(x) + u(x) = \cos(\pi x) \text{ on } (0,1), \quad \text{with } u(0) = u(1) = 0.$$



- Solution features layers of width $\mathcal{O}(\varepsilon)$ near $x = 0$ and $x = 1$.
- Away from layers $u \approx \cos(\pi x)$.

Example (Another reaction-diffusion equation)

$$-\varepsilon^2 u''(x) + u(x) = \sin(\pi x) \text{ on } (0,1), \quad \text{with } u(0) = u(1) = 0.$$



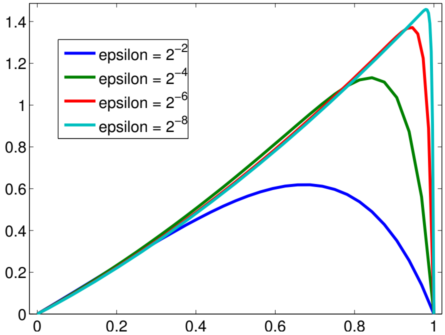
- The solution to the reduced equation satisfies the boundary conditions.
- The solution does not feature layers. In fact, $u = \sin(\pi x)/(\pi^2 \varepsilon^2 + 1)$.

Example (A convection-diffusion equation)

$$-\varepsilon u''(x) + u'(x) = x + 1 \text{ on } (0,1), \quad \text{with } u(0) = u(1) = 0.$$

The solution to this problem is

$$\frac{\varepsilon + 3/2}{1 - e^{-1/\varepsilon}}\left(e^{-1/\varepsilon} - e^{-(1-x)/\varepsilon}\right) + \varepsilon x + x^2/2 + x;$$



- Notice that the diffusion coefficient is ε, and not ε^2.
- The solution possesses a single layer, near $x = 1$.
- Elsewhere, the solution resembles that of

$$u' = x + 1.$$

- Computing stable solutions can be a challenge for this problem.

Coupled systems

The study of these simple-looking ODEs can rapidly become rather complex when extended to coupled systems. Our simplest example has just two equations.

Example (A coupled system of reaction-diffusion equations)

$$-\begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix}^2 \mathbf{u}'' + B(x)\mathbf{u} = \mathbf{f} \text{ on } (0,1), \quad \text{with } \mathbf{u}(0) = \mathbf{u}(1) = 0.$$

There are many variants possible for this problem, including

1. Systems of $\ell > 1$ equations;
2. Systems of convection-diffusion equations;
3. Strongly coupled systems;
4. ....

Example (A coupled system of reaction-diffusion equations)

$$-\begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix}^2 \mathbf{u}'' + \mathrm{B}(x)\mathbf{u} = \mathbf{f} \text{ on } (0,1), \quad \text{with } \mathbf{u}(0) = \mathbf{u}(1) = 0.$$

In spite of its simplicity, there is much that can be learned from this problem, which itself is often reduced to three sub-classes:

(a) $\varepsilon_1 = \varepsilon_2 \ll 1$

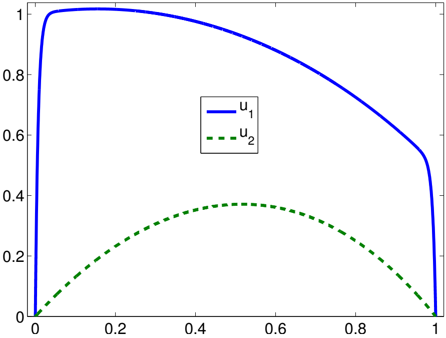
(b) $\varepsilon_1 \ll \varepsilon_2 = 1$

(c) $\varepsilon_1 \ll \varepsilon_2 \ll 1$

Case (a) is the least interesting. Under reasonable assumptions on B, most techniques (numerical and mathematical) for uncoupled problems extend directly to this case.

Example (Case (b): $\varepsilon_1 \ll \varepsilon_2 = 1$)

$$-\begin{pmatrix} 10^{-2} & 0 \\ 0 & 1 \end{pmatrix}^2 \mathbf{u}'' + \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 2-x \\ 1+e^x \end{pmatrix} \text{ on } (0,1), \quad \text{with } \mathbf{u}(0) = \mathbf{u}(1) = 0.$$

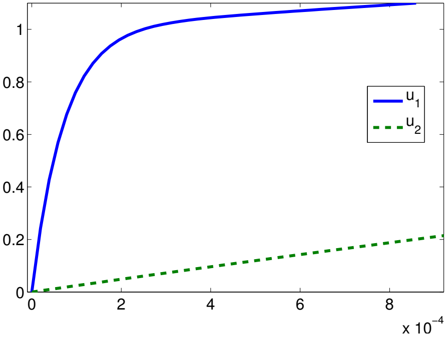


- The component u_1 features (strong) layers, of width $\mathcal{O}(\varepsilon)$.
- u_2 features "weak" layers: u_2' and u_2'' are bounded independent of ε, but u_2''' is not.

This is the most interesting case, since solutions possess multiple, interacting layers.

Example (Case (c): $\varepsilon_1 \ll \varepsilon_2 \ll 1$)

$$-\begin{pmatrix} 10^{-4} & 0 \\ 0 & 10^{-2} \end{pmatrix}^2 \mathbf{u}'' + \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 2-x \\ 1+e^x \end{pmatrix} \text{ on } (0,1), \quad \mathbf{u}(0) = \mathbf{u}(1) = 0.$$



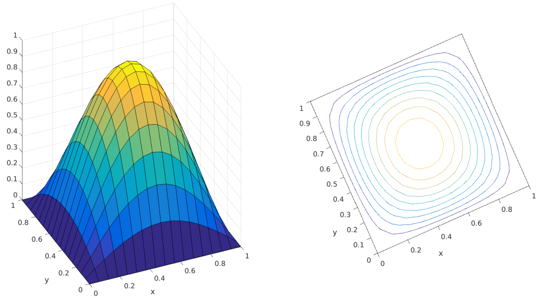
- Both components clearly features layers of width $\mathcal{O}(\varepsilon_2)$.
- u_1 also features a layer of width $\mathcal{O}(\varepsilon_1)$.

Reaction-diffusion PDEs

To orient ourselves, we begin by considering a regular problem.

A regular problem

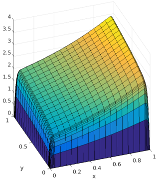
$$-(u_{xx} + u_{yy}) + b(x, y)u = f(x, y), \text{ on } (0, 1)^2 \qquad u = 0 \text{ on the boundary.}$$

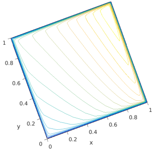


Next, we introduce the perturbation parameter.

A 2D singularly perturbed problem on the unit square

$$-\varepsilon^2(u_{xx} + u_{yy}) + b(x,y)u = f(x,y), \text{ on } (0,1)^2 \qquad u = 0 \text{ on the boundary.}$$

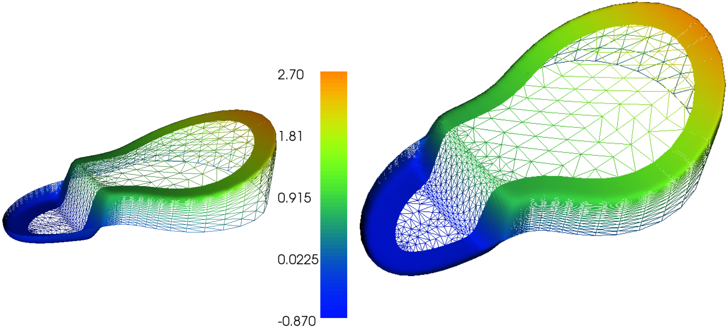




- Typically, on this domain, solutions feature four "edge" layers that behave like $\exp(-x/\varepsilon)$ or $\exp(-y/\varepsilon)$.
- They also have four corner layers, that behave like $\exp(-(x+y)/\varepsilon)$.

Reaction-diffusion PDEs

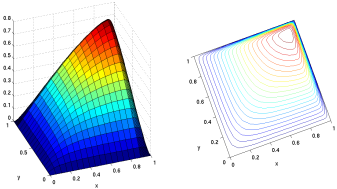
Of course, the problems don't have to be posed on the unit square. Doing so admits several interesting complications, though it also permits some simplification of the method and analysis..



(This example is based on one devised by Natalia Kopteva; see, e.g., [Kopteva & Pickett, 2012].)

A 2D singularly perturbed convection-diffusion problem

$$-\varepsilon(u_{xx}+u_{yy})+(u_x+u_y)=f(x,y), \text{ on } (0,1)^2 \qquad u=0 \text{ on the boundary.}$$



- The solution to this problem features layers near $x=1$ and $y=1$.
- All these layers are of width $\mathcal{O}(\varepsilon)$.
- The more general problem is

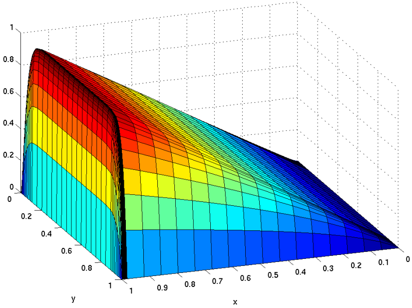
$$-\varepsilon\Delta u+\mathbf{a}\cdot\nabla u=f.$$

The location and width of the layers depend on $\mathbf{a}$.

Finally, we consider a problem for which flow is parallel to one of the axes.

Another 2D singularly perturbed convection-diffusion problem

$$-\varepsilon(u_{xx} + u_{yy}) + u_x = f(x, y), \text{ on } (0, 1)^2 \qquad u = 0 \text{ on the boundary.}$$



- The solution to this problem features three layers: near $x = 1$, $y = 0$, and $y = 1$.
- The ("exponential") layer near $x = 1$ is of width $\mathcal{O}(\varepsilon)$.
- The ("parabolic") layers near $y = \{0, 1\}$ are of width $\mathcal{O}(\sqrt{\varepsilon})$.

Other problems

There are many other important variants on the above problem, with the most obvious ones being:

1. Nonlinear problems.
2. Time dependent problems. Let L_ε be any of the differential operators considered above. Then solve
$$\frac{\partial u}{\partial t} + L_\varepsilon u = f(\cdot, t) \text{ on } \Omega \times (t_0, T].$$
3. Systems of first-order problems.
4. High-order problems, in particular 4th-order DEs.
5. PDEs in higher dimensions.

Slightly less obvious are important classes of SPDEs with solutions that feature *interior* layers. These can occur if, for example,

- ODES: Problem data that are discontinuous or result in turning points;
- ODEs: delay differential equations;
- PDEs: discontinuities in boundary data, which propagate as interior layers.
- PDEs: data incompatibilities.
- PDEs: irregular domains.

Discussion

For each of the above problems, away from the boundary, the *reduced* problem is obtained by setting $\varepsilon = 0$, and neglecting some or all of the boundary conditions. Layers then result when the solution to the reduced problem is reconciled with the missing boundary condition(s).

Therefore, for many problems, we can get a good sense of what the solution should look like by considering the reduced problem.

Discussion

References

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