Theory & Computation of Singularly Perturbed Differential Equations

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https://skumarmath.wordpress.com/gian-17/singular-perturbation-problems/

Niall Madden, NUI Galway

§3 Finite Difference Methods and their Analyses

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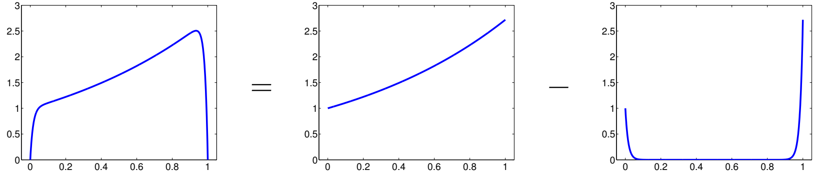


Outline

Monday, 4 December		
09:30 – 10.30	Registration and Inauguration	
10:45 – 11.45	1. Introduction to singularly perturbed problems	NM
12:00 – 13:00	2. Numerical methods and uniform convergence	NM
14:30 – 15:30	Tutorial (Convection diffusion problems)	NM
15:30 – 16:30	Lab 1 (Simple FEMs in MATLAB)	NM

Tuesday, 5 December		
09:30 – 10:30	3. Finite difference methods and their analyses	NM
10:45 – 11:45	4. Coupled systems of SPPDEs	NM
14:00 – 16:00	Lab 2 (Fitted mesh methods for ODEs)	NM

Thursday, 7 December		
09:00 – 10:00	8. Singularly perturbed elliptic PDEs	NM
10:15 – 11:15	9. Finite Elements in two and three dimensions	NM
01:15 – 15:15	Lab 4 (Singularly perturbed PDEs)	NM

Friday, 8 December		
09:00 – 10:00	10. Preconditioning for SPPs	NM

§3 Finite Difference Methods and their Analyses

(60 minutes)

Primary references

The main mathematical content of this presentation, starting at Slide 7, closely follows [Miller et al., 1996] and [Miller et al., 2012].

Important secondary references include [Protter and Weinberger, 1984].

Analysis

This section is devoted to the mathematical analysis of solutions to *one-dimensional reaction-diffusion equations*, and **finite difference** methods for approximating them.

Notation

The following notation applies in the remainder of this section.

- $\Omega := (0, 1)$.
- $\bar{\Omega}^N = \{0 = x_0 < x_1 < \cdots < x_N = 1\}$ is a (possibly arbitrary) mesh with N intervals.
  Ω^N denotes the interior of this mesh, i.e., $\Omega^N = \{x_1, \ldots, x_{N-1}\}$.
- $\|\cdot\|$ is the maximum norm on $\mathcal{C}(\bar{\Omega})$. That is $\|u\| := \|u\|_{\infty,\bar{\Omega}} = \max_{0\leqslant x\leqslant 1} |u(x)|$.
- $\|\cdot\|_{\bar{\Omega}^N}$ is the discrete max norm for mesh functions on Ω^N.

Always: C is a constant that is independent of ε and N. It can take different values in different places – even in the same expression.

Definition (Differential operator $\mathcal{L}$)

Let $\Omega := (0, 1)$. Given the function $b \in \mathcal{C}^4(\bar{\Omega})$, subject to $b(x) \geqslant \beta^2 > 0$, and parameter $\varepsilon \in (0, 1]$, the differential operator $\mathcal{L}$ is defined, for all $\psi \in \mathcal{C}^2(\bar{\Omega})$ as

$$\mathcal{L}\psi = -\varepsilon^2\psi'' + b\psi$$

Definition (Reaction-diffusion equation)

Let u be the solution to

$$\mathcal{L}u = f \text{ on } (0, 1), \text{ with boundary conditions } u(0) = u(1) = 0, \tag{1}$$

where $f \in \mathcal{C}^4(\bar{\Omega})$.

The imposition of homogeneous Dirichlet boundary conditions is only to simplify the exposition. Results are easily extended to more general situations.

Maximum principles

The following result is very standard, and elementary, but worth considering in detail in order to be able to generalise later.

Lemma

Maximum Principle If $\phi(0) \geqslant 0$, $\phi(1) \geqslant 0$, *and* $\mathcal{L}\psi(x) \geqslant 0$ *for all* $x \in \Omega$, *then* $\phi(x) \geqslant 0$ *for all* $x \in \bar{\Omega}$.

[*arguments on the board*]

Maximum principles

There are many useful consequences of this. For example:

It easily follows that, if $\mathcal{L}u = f$ and $u(0) = u(1) = 0$, then

$$|u(x)| \leqslant \|f\|/\beta^2 \qquad \text{for all } x \in \Omega.$$

[*arguments on the board*]

(Here $\psi = \|f\|/\beta^2$ is called a *barrier function*; we'll see more of these).

This shows that u is bounded. But for the analysis of a finite difference method, we need bounds on derivatives of u.

Bounds on derivatives

Lemma (Lemma 6.1 of [Miller et al., 2012])

$$\|u^{(k)}\| \leqslant C(1 + \varepsilon^{-k}).$$

[*arguments on the board*]

Bounds on derivatives

The above bounds are correct, but not at all sharp: one expects that $|u''(x)| \approx \varepsilon^{-2}$ only near the boundary.

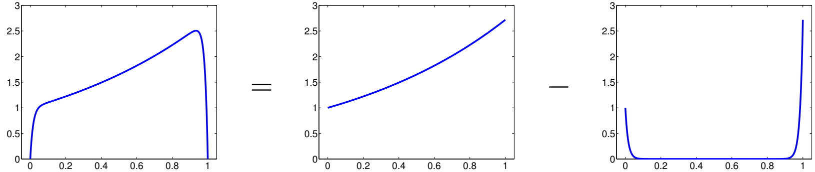
Our numerical analysis will require sharper, point-wise bounds obtained via a *decomposition* of the solution of the DE as the sum of "regular" and "layer" components:

$$u = \underbrace{v}_{\text{regular}} + \underbrace{w}_{\text{layer}}.$$

Solution decomposition

Solution decomposition

$$u = \underbrace{v}_{\text{regular}} + \underbrace{w}_{\text{layer}}.$$



Roughly,

- v represents the solution away from the boundaries, but any layers present are only weakly expressed.
- w accounts for the boundary conditions and, thus, the boundary layers, but decays rapidly away from the boundaries.

Definition (Reduced problem)

To get the reduced problem, set $\varepsilon = 0$ in the reaction-diffusion equation (1), and neglect the boundary conditions. That is, let v_0 be the solution to

$$b(x)v_0(x) = f(x).$$

Regular part

Let $v = v_0 + \varepsilon^2 v_1$, where v_0 is the reduced solution, and v_1 solves

$$\mathcal{L}v_1 = v_0'', \qquad v_1(0) = v_1(1) = 0.$$

From our earlier lemma, it is clear that $|v_1^{(k)}(x)| \leqslant C(1 + \varepsilon^{-k})$.
It follows that $|v^{(k)}(x)| \leqslant C(1 + \varepsilon^{-k+2})$.

Layer part

Since $u = w + v$, we have that w satisfies the homogeneous DE

$$-\varepsilon^2 w'' + bw = 0 \text{ on } \Omega, \qquad w = u - v \text{ on } \partial\Omega$$

Define the boundary layer function

$$\mathcal{B}_\varepsilon(x) := \exp(-x\beta/\varepsilon) + \exp(-(1-x)\beta/\varepsilon).$$

Also define the barrier function

$$\psi^{\pm}(x) = C\mathcal{B}_\varepsilon(x) \pm w(x)$$

for some suitable large C so that $\psi^{\pm}$ is non-negative on the boundary. Then the Maximum Principle shows that $|w(x)| \leqslant \mathcal{B}_\varepsilon(x)$ on $\bar{\Omega}$.

Now a minor variant on the argument that was used to bound $|u^{(k)}|$ will give that

$$|w^{(k)}(x)| \leqslant C\varepsilon^{-k}\mathcal{B}_\varepsilon(x).$$

The above solution decomposition tells use that the derivatives of u *are* large, but decay rapidly away from the boundaries. So it will be of no surprise to learn that there is a good strategy for solving these problems that involves a mesh condensing near those boundaries.

First, for an arbitrary mesh $\Omega^N = \{x_0, x_1, \dots, x_n\}$ the standard second-order finite-difference operator becomes

$$(\delta^2 v)_i = \frac{2}{h_{i+1} + h_i} \left(\frac{v_{i+1} - v_i}{h_{i+1}} - \frac{v_i - v_{i-1}}{h_i} \right).$$

where $h_i = x_i - x_{i-1}$.

It satisfies

E.g., [Miller et al., 2012, Lemma 4.1]

$$\left| (\delta^2 - \frac{d^2}{dx^2}) \phi(x_i) \right| \leqslant \frac{1}{3} (x_{i+1} - x_{i-1}) |\phi|_3.$$

Then the FDM is

$$U_0 = 0,$$
$$L^N \mathbf{U}_i := -\varepsilon^2 \delta^2 U_i + b(x_i) U_i = f(x_i), \qquad i = 1, \dots, N-1,$$
$$U_N = 0.$$

This operator satisfies a *discrete maximum principle*, and an ε-uniform stability result: if $\Phi_0 \geqslant 0$, $\Phi_N \geqslant 0$, and $L^N \Phi_i \geqslant 0$, then

$$|\Phi_i| \leqslant \frac{1}{\beta} \max_{i \leqslant j \leqslant N} |L^N \Phi_j|.$$

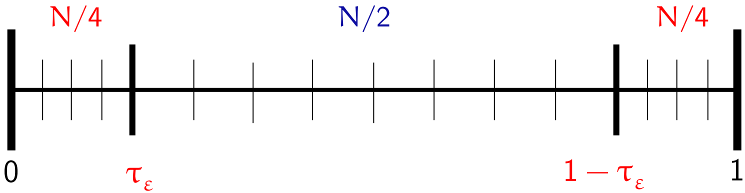
[*arguments on the board*]

The piecewise uniform mesh of *Shishkin* is constructed as follows[1]:

1. Choose a **mesh transition point**

$$\tau_\varepsilon = \min\left\{\frac{1}{4}, \frac{\varepsilon}{\beta} \ln N\right\}.$$

2. Divide the domain into three sub-regions: $[0, \tau_\varepsilon]$, $[\tau_\varepsilon, 1 - \tau_\varepsilon]$ and $[1 - \tau_\varepsilon, 1]$.
3. Subdivide those sub-regions to obtain the mesh.



[1] This choice is for the purpose of exposition. It is better to choose $\tau_\varepsilon = \min\{1/4, 2\varepsilon/\beta \ln N\}$.

Theorem (Theorem 6.4 of [Miller et al., 1996])

There is a constant C *that is independent of* ε *and* N *such that*

$$\|u - U\|_{\bar{\Omega}^N} \leqslant CN^{-1} \ln N.$$

The proof proceeds by constructing a decomposition of the discrete solution $U = V + W$ that is analogous to the decomposition of the continuous solution:

$$\begin{array}{ll} V \text{ solves} & V_0 = v(0), \quad L^N V_i = f(x_i), \quad V_N = v(1), \\ W \text{ solves} & W_0 = w(0), \quad L^N W_i = 0, \quad W_N = w(1). \end{array}$$

We analyse these terms separately, i.e., we estimate $\|v - V\|$ and $\|w - W\|$.
For $\|v - V\|$... [*arguments on the board*]

Analysis

On the region $(0, \tau_\varepsilon) \cup (1 - \tau_\varepsilon, 1)$ the analysis for $\|w - W\|$ is analogous...
[*arguments on the board*]

Analysis

Finally, on the region $[\tau_\varepsilon, 1-\tau_\varepsilon]$ one can exploit the fact that w and W have decayed. In particular, for any $x_i \in [\tau, 1-\tau]$

$$\mathcal{B}_\varepsilon(x_i) \leqslant \mathcal{B}_\varepsilon(\tau) \leqslant 2\exp(-\tau\beta/\varepsilon) \leqslant 2\exp(-\ln N) \leqslant CN^{-1}.$$

[*arguments on the board*]

Wrap up

- The significance of the above result is that we have designed a scheme for which we can prove that the pointwise error is independent of ε.

- The result is easily extended to show that $\|u - \bar{U}\| \leqslant CN^{-1} \ln N$.

- The result is correct, but not sharp. Can the scheme and analysis be improved so that $\|u - U\|_{\bar{\Omega}^N} \leqslant CN^{-2} \ln^2 N$? [Discuss!]

- The approach here, of using a piecewise uniform mesh, is very elementary. A more sophisticated mesh, such as the graded mesh of Bakhvalov, can yield a fully second-order scheme.

References

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