On the application of robust numerical methods to a complete-flow wave-current model

Niall Madden[†], Martin Stynes[‡], and Gareth Thomas[♭]

[†]Department of Mathematics, National University of Ireland, Galway. Ireland.
Niall.Madden@NUIGalway.ie.
[‡]Department of Mathematics, National University of Ireland, Cork, Ireland.
m.stynes@ucc.ie
[♭]Department of Applied Mathematics, University College Cork, Ireland.
G.Thomas@ucc.ie

Abstract

It will be shown how parameter-robust numerical methods can be used to solve equations that arise in the modelling of wave-current interactions. Two such models are presented: a complete flow model for wave-current interaction in the presence of weakly turbulent flow leading to an Orr-Sommerfeld type problem and a system of two singularly perturbed reaction-diffusion equations from a k-ϵ turbulence model. The numerical results are compared with experimental data.

1. Introduction

Parameter-robust numerical methods are of significant interest in modern numerical analysis: they yield accurate, layer-resolved, computed solutions to singularly perturbed differential equations. Importantly, their accuracy is independent of the singular perturbation parameter and thus the width of the boundary layers.

Many of these methods are *mesh based*. That is, they use the same discretizations as one would use for a *classical* problem whose solution does not exhibit layers. Instead of modifying the scheme to stabilize it, a mesh tailored to the specific problem is used. In this study we employ the *a priori* fitted piecewise-uniform meshes of Shishkin, as described in [9].

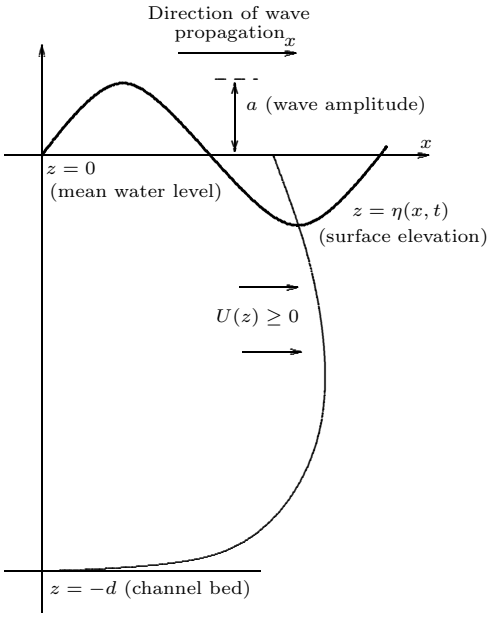
A complete flow model for the interaction of waves and currents leads to a variant on the fourth-order Orr-Sommerfeld equation, and is described in §2. A crucial component of the model is a depth-varying eddy viscosity distribution. For a given current profile, this is computed using a two-equation turbulence model described in §4.

Since the models generate the function that establishes the width of the boundary layers, it is appropriate to use a numerical method whose accuracy is independent of the layer width.

2. A complete flow model

The physical system consists of regular waves of frequency ω and local amplitude a propagating along a channel of uniform depth d. It is usual to take the origin of the co-ordinate system to lie in the *mean water level*, with the positive x-axis aligned with the direction of wave propagation and z is measured positive in a vertically upwards direction. Boundary conditions are imposed at the mean free surface $z = 0$ and at the channel bed $z = -d$.

From: - Proceedings of International Conference on Boundary and Interior Layers — Computational and Asymptotic Methods 2004, Toulouse, France. [Proceedings article]



A steady current $U(z)$ is also present; it is also assumed to flow in the x-direction but may possess vertical variation. This represents the horizontal velocity of a fluid particle in the absence of waves; if waves are present, the current denotes the horizontal velocity averaged over a wave cycle. No distinction is made between components in the current due to externally-imposed flows and the mean flow induced by the wave motion. In the accompanying diagram, $U(z)$ is shown acting in the same direction as the waves and is then called a *wave-following current*. Alternatively, we may consider a *wave-opposing current*. It is assumed that $U(z)$ is known and so is said to be of *reactive* type [7]. Some models attempt to predict $U(z)$, these are sometimes referred to as *generative* [1, 2]).

Three important assumptions are made to enable the construction of a suitable model. Firstly, the slope of the waves is considered to be small and that a good approximation to the wavelike motion can be obtained by consideration of only the first-order sinusoidal terms. Secondly, the fluid temperature is considered to be uniform, so that the molecular viscosity ν is uniform throughout the flow. Thirdly, a key aspect of the model we present here is the use of an *eddy viscosity distribution* to model the turbulent characteristics of the flow. The Boussinesq eddy viscosity concept assumes that there is an analogy between turbulence stress and viscous stresses in laminar flow. For our purposes it can be taken to mean that there is a depth-varying non-negative quantity, the (kinematic) eddy viscosity $\nu_t(z)$, that mimics the dynamic behaviour of the kinematic molecular viscosity ν. Therefore the model (1–5) below is formulated so that the term ν is augmented by $\nu_t(z)$.

The local wave amplitude a and frequency ω are taken as known and used as inputs to the model, in addition to the current field $U(z)$. The intended outputs are the complex wavenumber k and the wavelike velocity fields. It is straightforward to interpret the real and imaginary parts of k: the real part linked to the wavenumber λ and the imaginary part determines the spatial wave decay due to dissipation.

Due to the interaction of waves with the steady current, both the total horizontal and vertical velocities of a particle, $u_T(x, z, t)$ and $w_T(x, z, t)$, do indeed vary with time. These can be written as

$$u_T(x, z, t) = U(z) + u_w(x, z, t), \qquad w_T(x, z, t) = w_w(x, z, t),$$

where the time average, per wave cycle, of terms with a w-subscript is zero. It is usual to introduce a stream-function $\Psi(x, z, t)$, which is related to the wavelike velocity components by

$$u_w(z) = \frac{\partial \Psi}{\partial z}, \qquad w_w(z) = -\frac{\partial \Psi}{\partial x}.$$

As we intend to compute approximations to the first order velocity components only, it is justifiable to assume that $\Psi(x, z, t)$ is of the form

$$\Psi(x, z, t) = \Re\left\{\psi(z) \exp(i\theta)\right\},$$

where $\theta = kx - \omega t$ is the phase function and $\Re\{\cdot\}$ denotes the real part of a complex number.

The differential equation that must be solved for ψ and k is the fourth-order boundary-value problem

$$-\frac{i\nu}{\omega-kU}\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-k^2\right)^2\psi+\frac{\mathrm{d}\psi^2}{\mathrm{d}z^2}-\left(k^2-\frac{k}{\omega-kU}\frac{\mathrm{d}^2U}{\mathrm{d}z^2}\right)\psi=$$
$$\frac{i}{\omega-kU}\left\{\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}+k^2\right)\left\{\nu_t\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}+k^2\right)\psi\right\}-4k^2\frac{\mathrm{d}}{\mathrm{d}z}\left(\nu_t\frac{\mathrm{d}\psi}{\mathrm{d}z}\right)\right\}\quad\text{on }\Omega:=(-d,0),\quad(1)$$

subject to the bottom boundary conditions

$$\psi=0,\quad\frac{\mathrm{d}\psi}{\mathrm{d}z}=0,\qquad\text{on } z=-d.\quad(2)$$

The three free surface ($z=0$) conditions are

$$\psi=a\left(\frac{\omega}{k}-U\right),\quad(3)$$
$$(\nu+\nu_t)\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2}=-a\left((\nu+\nu_t)\frac{\mathrm{d}^2U}{\mathrm{d}z^2}+\frac{\mathrm{d}\nu_t}{\mathrm{d}z}\frac{\mathrm{d}U}{\mathrm{d}z}+k(\nu+\nu_t)(\omega-kU)\right),\quad(4)$$

and

$$i(\nu+\nu_t)\frac{\mathrm{d}^3\psi}{\mathrm{d}z^3}+i\frac{\mathrm{d}\nu_t}{\mathrm{d}z}\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2}-\left\{\omega-kU+3ik^2(\nu+\nu_t)\right\}\frac{\mathrm{d}\psi}{\mathrm{d}z}=a\left((\omega-kU)\left\{\frac{\mathrm{d}U}{\mathrm{d}z}-ik\frac{\mathrm{d}\nu_t}{\mathrm{d}z}\right\}-gk\right).\quad(5)$$

3. Numerical Solution of the Orr-Sommerfeld Equations

The equation (1) is singularly perturbed because the quantity $i(\nu+\nu_t(z))/(\omega-\kappa U)$ that multiplies the highest-order derivative $\psi^{(4)}(z)$ is small in magnitude. One of the consequences of this is that the second derivative of the solution at the bottom boundary ($z=-d$) will be large.

A similar problem, though real-valued and linear, was studied by Sun and Stynes [12]. They showed how to construct a piecewise uniform mesh on which one can apply a finite element method. The solution of the discrete problem converges to the continuous solution as the number of mesh points increases, and this convergence is independent of the magnitude of the coefficient of the highest-order derivative.

Our finite element formulation is based on the sesquilinear form

$$B_{(\nu+\nu_t)}(u,v):=((\nu+\nu_t)u'',v'')+k^2\left((\nu+\nu_t)u,v''\right)+4k^2\left((\nu+\nu_t)u',v'\right)$$
$$+\left([i(\omega-kU)+k^2(\nu+\nu_t)]u'',v\right)+\left(\left[ikU''-ik^2(\omega-kU)+k^4(\nu+\nu_t)\right]u,v\right),\quad(6)$$

for all $u,v\in H^2_\star(\Omega)$, where $H^2_\star(\Omega)$ is a subspace of $H^2(\Omega)$ chosen so that functions belonging to it satisfy the boundary conditions (2)–(3).

We seek not only an approximation of $\psi(z)$ but also an approximation of $\psi'(z)$ from which to predict the vertical component $w(z)$ of the flow velocity. Thus it is natural to choose piecewise cubic Hermite basis functions, which lie in $C^1(\bar{\Omega})$:

$$v_h(z)=\sum_{i=1}^{N}v_i\phi_i^0(z)+\sum_{i=1}^{N}v_i'\phi_i^1(z),$$

where $\phi_i^0(z)$ and $\phi_i^1(z)$ are the usual basic functions for Hermite interpolation.

An iterative scheme is then formulated based on boundary condition (5). Now the problem is to compute $\psi(x)$, $\psi'(x)$ and k such that the variational formulation of (1)–(4) and the extra condition (5) are satisfied. For further details of the numerical method, see [6] and [7].

4. The Turbulence Model

To calculate a suitable eddy viscosity distribution $\nu_t(z)$ for a given current profile, we employ a model due to Thomas [13]. This is a so-called *two equation model* based upon the well-known k-ϵ model, where k is

the kinematic energy per unit mass of the turbulent motion, and ϵ is the rate of viscous dissipation [10]. In the study of waves and currents, the notations k and ϵ are usually reserved for the wavenumber and wave-slope respectively. Therefore we follow Thomas's notation and use E for the turbulent kinetic energy and D for the rate of dissipation.

The approach of Thomas [13] is to expand $\nu_t(z)$, $E(z)$ and $D(z)$ in terms of the wave slope ϵ. Here we concentrate only on the zero-order equations, associated with the fundamental properties of the current. Our unknowns are then the functions $\nu_t^0(z)$, $E_0(z)$ and $D_0(z)$, which are the zero-order terms in the expansions of $\nu_t(z)$, $E(z)$ and $D(z)$. These quantities satisfy the three coupled nonlinear equations

$$\nu_t^0(z)D_0(z) = C_\mu f_\mu(z) E_0^2(z) \qquad \text{for } z \in \bar{\Omega}, \tag{7a}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\left(\nu + \frac{\nu_t^0}{\sigma_E}\right)\frac{\mathrm{d}E_0}{\mathrm{d}z}\right] + \nu_t^0(z)\left[\frac{\mathrm{d}U}{\mathrm{d}z}\right]^2 = D_0(z) \qquad \text{for } z \in \Omega, \tag{7b}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\left(\nu + \frac{\nu_t^0}{\sigma_D}\right)\frac{\mathrm{d}D_0}{\mathrm{d}z}\right] + C_{1D} f_1(z) C_\mu f_\mu(z) E_0 \left[\frac{\mathrm{d}U}{\mathrm{d}z}\right]^2 = C_{2D} f_2(z) \frac{D_0^2}{E_0} \qquad \text{for } z \in \Omega, \tag{7c}$$

where $\Omega := (-d, 0)$. The boundary conditions are

$$E_0(-d) = E'(0) = D_0'(-d) = 0, \quad D_0(0) = (5.87/d) E_0^{\frac{3}{2}}(0). \tag{8}$$

The three functions $f_\mu(z)$, $f_1(z)$ and $f_2(z)$ in (7) are *wall functions* and are present in the formulation to ensure correct behaviour in the near-wall region. In general their value is close to unity over all of Ω, except near the bottom boundary where they change rapidly. For details of the these functions and the values of the empirically established terms C_{1D}, C_{2D} (closure constants) and σ_E and σ_D (diffusion constants) we refer the reader to [4].

The E-D equations are solved numerically by applying a finite element method on a piecewise uniform mesh. Such meshes for systems of reaction-diffusion equations have been analysed by Shishkin and collaborators, see e.g., [11]. Solutions to the system that we study here exhibit two distinct, interacting layers near each boundary. It has been shown that a piecewise uniform mesh can yield a parameter-robust approximation with finite difference [8] and finite element [5] methods.

To resolve the interacting layers at the bed, we construct a piecewise uniform mesh with two transition parameters τ_1 and τ_2 chosen according to the formula

$$\tau_1 = -d + C_1 \nu \ln N \quad \text{and} \quad \tau_2 = -d + C_2 \sqrt{\nu} \ln N,$$

where C_1 and C_2 are user-chosen values. Thus we divide $[-d, 0]$ into the three subintervals $[-d, \tau_1]$, $[\tau_1, \tau_2]$, and $[\tau_2, 0]$. We place a uniform mesh on each subinterval in such a way that there are $N/4$ evenly spaced mesh points on each of the subintervals $[-d, \tau_1]$, $[\tau_1, \tau_2]$, and $N/2$ mesh points in the remainder of the region.

5. Numerical Results

We now compare numerical results with data from physical experiments of Klopman [3], where waves of local amplitude $a = 0.05987$m and frequency $\omega = 0.9844$ Hz were propagated in a flume with water of depth 0.5m.

The current distribution $U(z)$ for the wave-following case is shown on the left of Figure 1. The data is fitted with a continuous function which is then used as an input for the E–D model. The resulting $\nu_t^0(z)$ is shown in the centre of Figure 1. A detail of the $\nu_t^0(z)$ close to the channel bed is shown on the right.

Figure 2 below show the predicted horizontal component of the orbital velocity. We give both the results obtained when $\nu_t^0(z) \equiv 0$ (i.e., we neglect the effects of turbulence) and with $\nu_t^0(z)$ as given in Figure 1. From the picture on the left we see that both approaches yield predictions that agree very well

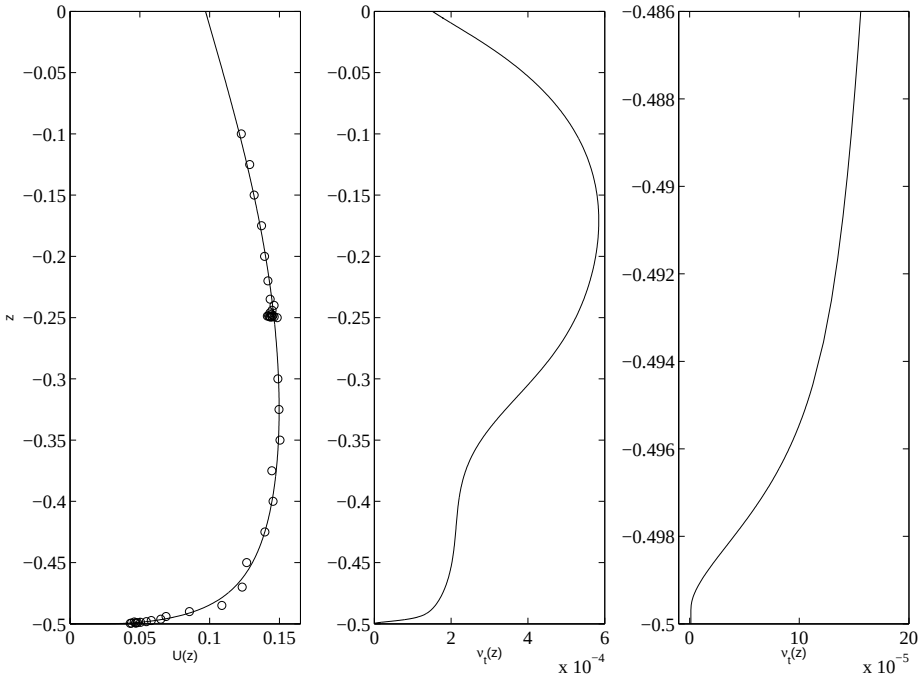


Figure 1: A current profile $U(z)$ and computed $\nu_t^0(z)$

(over most of the depth) with the experimentally obtained data. In the picture on the right we show the results obtained in the region closest to the channel bed. Here the model for pure viscous flow clearly under estimates the width of the boundary layer. When the eddy viscosity term in included, it seems that the layer is excessively dissipated. The computed wave numbers were $k = 2.149 + 4.509 \times 10^{-4} i$ for $\nu_t^0 \equiv 0$, and $k = 2.149 + 3.230 \times 10^{-3} i$ when the turbulence term is included.

6. Conclusion

We have shown that piecewise uniform meshes can be successfully applied to the study of wave-current interactions.

Within the region closest to the bed, the agreement between experimental data and the numerical simulation of the model is not entirely satisfactory. However, careful selection of the problem parameters, data parameterization, and choice of wave-functions is an area of on-going research.

References

[1] J. Groeneweg, *Wave-current interactions in a generalized Lagrangian mean formulation*, PhD thesis, Technische Universiteit Delft, 1999.

[2] J. Groeneweg and G. Klopman, *Changes of the mean velocity profiles in the combined wave-current motion described in a GLM formulation.*, J. Fluid Mech., 370 (1998), pp. 271–296.

[3] G. Klopman, *Vertical structure of the flow due to waves and currents*, Progress Report H840.30, Part 4, Delft Hydraulics, 1994.

[4] C. K. G. Lam and K. Bremhorst, *A modified for of the k-ε model for predicting wall turbulence*, Trans. ASME, 103 (1981), pp. 456–460.

[5] T. Linß and N. Madden. A finite element analysis of a coupled system of singularly perturbed reaction-diffusion equations. Appl. Math. Comp. To appear.

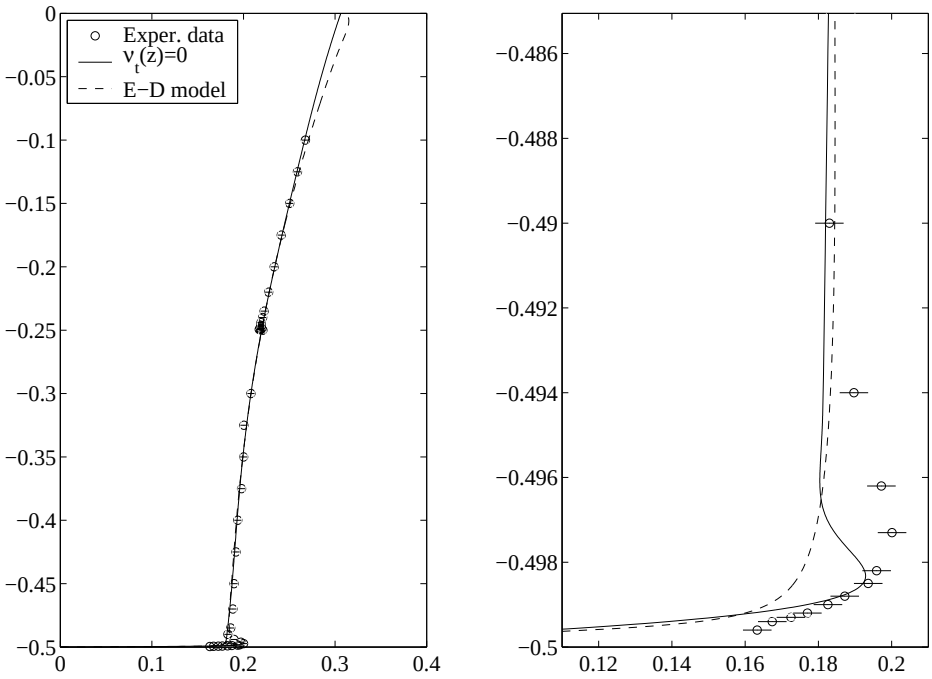


Figure 2: Prediction of the horizontal component of the vertical orbital velocity

[6] N. Madden, "Numerical Methods for Wave-Current Interactions", Ph.D. Thesis, NUI Cork. 2000.

[7] N. Madden, M. Stynes, and G. P. Thomas, "0n the development of complete flow models for wave-current interactions", *Proc. Coastal Dynamics 97, ASCE, Plymouth, United Kingdom*, 1997.

[8] N. Madden and M. Stynes, " A uniformly convergent numerical method for a coupled system of two singularly perturbed linear reaction-diffusion problems". IMA Journal of Numerical Analysis, **23**, 627-644 (2003).

[9] J. J. H. Miller, E. O'Riordan, and G. I. Shishkin, *Fitted numerical methods for singular perturbation problems – error estimates in the maximum norm for linear problems in one and two dimensions*, ISBN 981-02-2462-1, World Scientific, 1996.

[10] W. Rodi, *Turbulence Models and Their Applications in Hydraulics*, IAHR Monograph Series, A. A. Balkema, Rotterdam, 3rd ed., 1993.

[11] G. I. Shishkin, *Mesh approximation of singularly perturbed boundary-value problems for systems of elliptic and parabolic equations*, Comp. Maths. Math. Phys., 35 (1995), pp. 429–446.

[12] G. Sun and M. Stynes, *Finite-element methods for singularly perturbed high-order elliptic two-point boundary value problems I: reaction–diffusion-type problems*, IMA J. Numer. Anal., 15 (1995), pp. 117–139.

[13] G. P. Thomas, Towards an improved turbulence model for wave-current interactions. In *Second Annual Report to EU MAST-III Project* The Kinematics and Dynamics of Wave-Current Interactions. Contract No. MAS3-CT95-0011, 14 pages, 1998.