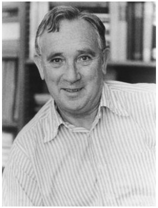
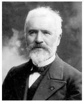
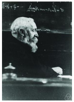
Cholesky, Taussky, Todd, Toeplitz and Turing: A story of love, death, wars, and matrix factorisations

NUI Galway Mathematics Society, 25 Nov 2011





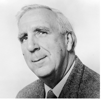
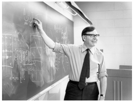








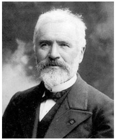


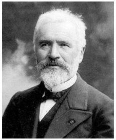

High & Low Land
Over 5,000 Ft.
1500 to 5,000
700 to 1,500
0 to 700
ATLANTIC OCEAN
British Isles
Ireland
Great Britain
Hebrides
Faroe Is
Shetland Is
Orkney Is
C. Wrath
Land's End
English Channel
C. Ushant
St. of Dover
North Sea
The Naze
Skager Rack
Cattegat
Galdhopiggen 8400
Scandinavia
G. of Bothnia
G. of Finland
G. of Riga
Gothland
Baltic Sea
L. Ladoga
L. Onega
R. Dwina
L. Peipus
Valdai Hills 1200
R. Volga
Plain of Europe
R. Kama
Konjakofski 5200
Yaman Tau 5500
Tobol
R. Ural
R. Duna
R. Oka
R. Don
R. Dnieper
R. Dniester
Volga
Caspian Sea
Central
R. Elbe
R. Oder
R. Vistula
Rhine
R. Seine
R. Loire
Bay of Biscay
C. Ortegal
C. Finisterre
Mts of Auvergne
Cevennes Mts
Mont Blanc
The Alps
Danube
Mt Tatra 8700
Carpathian Mts
Plain of Hungary
Plain of Wallachia
R. Drave
R. Save
Pyrenees Mts
Mt Maladetta
Iberian Pena
R. Douro
R. Tagus
R. Guadiana
R. Guadalquivir
Sierra Nevada
C. Roca
C. St. Vincent
C. Tarifa
St. of Gibraltar
G. of Lions
G. of Genoa
Corsica
St. of Bonifacio
Sardinia
Balearic Is
Apennines
Adriatic Sea
Tyrrhenian Sea
Mt Vesuvius
Sicily
Mt Etna
Malta
G. of Taranto
Ionian Sea
Mt Olympus 9500
Schar Dagh 10000
Balkan Mts
Balkan Pena
Morea
C. Matapan
Aegean Sea
Crete (Candia)
MEDITERRANEAN SEA
Crimea
Sea of Azov
St. of Yenikale
Black Sea
Bosphorus
Sea of Marmora
Caucasus Mts
Elbruz 18500
Armenia
Mt Ararat 17000
L. Van
L. Urumiah
Asia Minor
Taurus Mts
Cyprus
Syria
Mesopotamia
R. Tigris
R. Euphrates
Little Atlas Mts
Great Atlas Mts
Morocco
Algeria
Tunis
G. of Gabes
Shott el Jerid

Andre-Louis Cholesky,
15 Oct 1875 – 31 Aug 1918

Andre-Louis Cholesky,
15 Oct 1875 – 31 Aug 1918



Andre-Louis Cholesky,
15 Oct 1875 – 31 Aug 1918



$$Mx = b \qquad M \in \mathbb{R}^{m \times n}, m > n$$

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \\ m_{41} & m_{42} & m_{43} \\ m_{51} & m_{52} & m_{53} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix}$$

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$$r = b - Mx.$$

Find x that minimises $\|r\|_2 = \sqrt{r^T r} \iff M^T M x = M^T b,$

Because:

$$0 = \lim_{e \to 0} \frac{\left(M(x+e) - b\right)^T \left(M(x+e) - b\right) - \left(Mx - b\right)^T \left(Mx - b\right)}{\|e\|}$$

$$= \lim_{e \to 0} \frac{2e^T \left(M^T M x - M^T b\right) + e^T M^T M e}{\|e\|}$$

$$A = M^T M \iff \begin{cases} \text{A is symmetric positive definite (spd)} \\ x^T A x > 0 \ \forall x \in \mathbb{R}^{n \times n}. \\ Ax = \lambda x \implies \lambda > 0. \\ f(x) = \frac{1}{2} x^T A x - x^T b \text{ is strictly convex for all } b. \\ A^{(k)} = A_{1:k,1:k}, k = 1, 2, \ldots, n \text{ is spd} \end{cases}$$

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$$A = \begin{pmatrix} a_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix} = \begin{pmatrix} \sqrt{a_{11}} & 0 \\ \frac{A_{12}^T}{\sqrt{a_{11}}} & \tilde{A}_{22} \end{pmatrix} \begin{pmatrix} \sqrt{a_{11}} & \frac{A_{12}}{\sqrt{a_{11}}} \\ 0 & \tilde{A}_{22} \end{pmatrix}$$

A. Cholesky

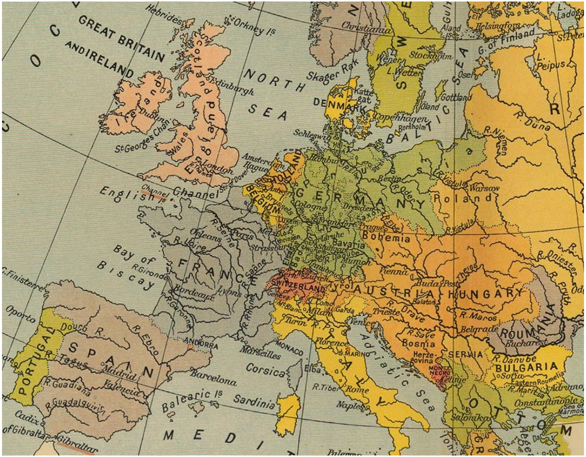
Sur la résolution numérique des Systèmes d'équations linéaires.

La solution des problèmes dépendant de données expérimentales, qui peuvent dans certains cas être soumises à des conditions, et auxquelles on applique la méthode des moindres carrés est toujours subordonnée au calcul numérique des racines d'un système d'équations linéaires. C'est le cas de la recherche des lois physiques ; c'est aussi le cas de la compensation des réseaux géodésiques. Il est donc intéressant de rechercher un moyen ~~sûr~~ et aussi simple que possible d'effectuer la résolution numérique d'un système d'équations linéaires.

Le procédé que nous allons indiquer s'applique aux systèmes d'équations symétriques auxquels conduit l'application méthode des moindres carrés, mais nous remarquerons ~~tout~~ d'abord que la résolution d'un système de n équations linéaires à n inconnues peut ~~très~~ facilement se ramener à la résolution d'un système ~~[illegible]~~ de n équations linéaires symétrique à n inconnues.

Considérons en effet le système suivant :

$$\mathrm{I}\quad\begin{cases} \alpha_1^1\gamma_1 + \alpha_2^1\gamma_2 + \alpha_3^1\gamma_3 + \cdots + \alpha_n^1\gamma_n + C_1 = 0 \\ \alpha_1^2\gamma_1 + \alpha_2^2\gamma_2 + \alpha_3^2\gamma_3 + \cdots + \alpha_n^2\gamma_n + C_2 = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_1^n\gamma_1 + \alpha_2^n\gamma_2 + \cdots\cdots + \alpha_n^n\gamma_n + C_n = 0 \end{cases}$$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Copenhagen
Bornholm
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G. of Finland
L. Ladoga
L. Peipus
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Biscay
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R. Saone
C. Finisterre
SWITZERLAND
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L. Como
L. Garda
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Vienna
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Balaton
R. Theiss
R. Drave
R. Maros
Trieste
Milan
Turin
Venice
Florence
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ITALY
Adriatic Sea
Rome
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Naples
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Corsica
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Palermo
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Barcelona
Balearic Is
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SERVIA
Belgrade
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R. Danube
BULGARIA
Sofia
Eastern Roumelia
R. Maritsa
Adrianople
Constantinople
Sea of Marmora
Salonika
OTTOM
R. Dniester
R. Pruth
Odessa
MEDI

Olga Taussky,
30 Aug 1906 – 7 Oct 1995

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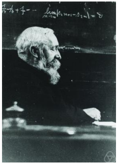
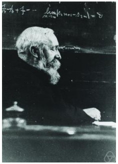


Olga Taussky,
30 Aug 1906 – 7 Oct 1995



Olga Taussky,

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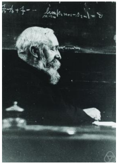

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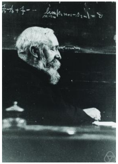

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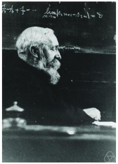

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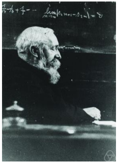

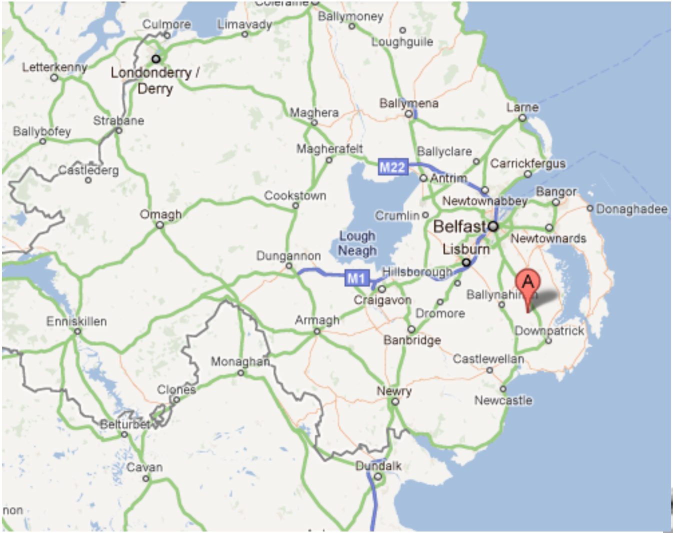
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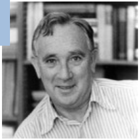
Ballymoney
Loughguile
Letterkenny
Londonderry /
Derry
Maghera
Ballymena
Larne
Strabane
Ballybofey
Magherafelt
Ballyclare
Carrickfergus
M22
Antrim
Castlederg
Cookstown
Newtownabbey
Bangor
Donaghadee
Omagh
Crumlin
Belfast
Lough
Neagh
Newtownards
Lisburn
Dungannon
M1
Hillsborough
A
Craigavon
Dromore
Enniskillen
Armagh
Downpatrick
Banbridge
Castlewellan
Monaghan
Clones
Newry
Newcastle
Belturbet
Dundalk
Cavan



John ("Jack") Todd,
16 May 2011 – 21 June 2007

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16 May 2011 – 21 June 2007

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16 May 2011 – 21 June 2007







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16 May 2011 – 21 June 2007








Matrices with finite period

By Olga Taussky and John Todd.

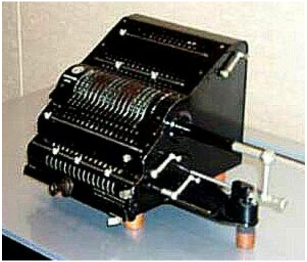
(*Received* 8*th September*, 1939. *Read* 25*th November*, 1939.)

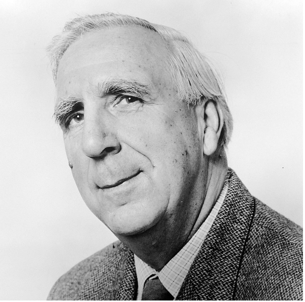
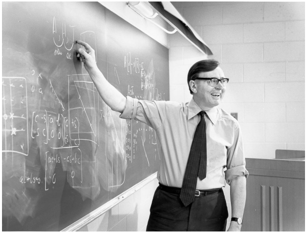
1. The study of the primitive solutions of the equation

$$A^r = 1, \tag{1}$$

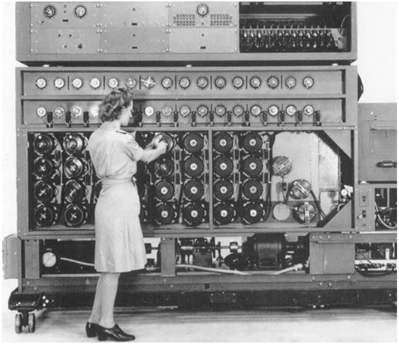
where $A = (a_{ij})$ is an $n \times n$ matrix whose elements are rational integers, was begun a long time ago[1]. In most cases this equation occurred incidentally in another theory; for instance Jordan encountered it in connection with linear differential equations having algebraic solutions, Minkowski in connection with quadratic forms and Turnbull in geometry. An important fact about these matrices is that any unimodular matrix can be represented as the product of matrices with finite period.

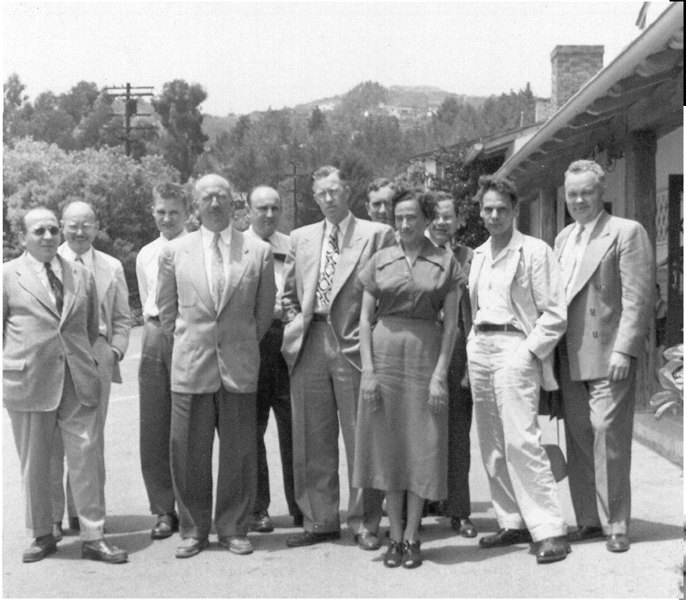
In this note we point out a connection between (1) and algebraic number fields generated by roots of unity. The methods we adopt can be applied to the study of a much larger class of matrices. In § 2 we summarise those parts of Algebraic Number Theory which we require[2].

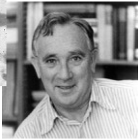



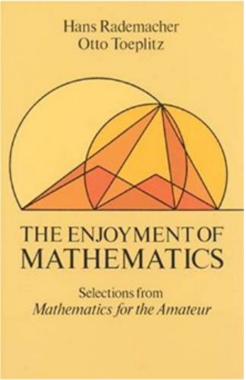

Alan Turing, 23 Jun 1912 – 7 Jun 1954







Otto Toeplitz
1 Aug 1881 – 15 Feb 1940



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Cholesky, Toeplitz and the triangular factorization of symmetric matrices

Olga Taussky and John Todd

California Institute of Technology, Pasadena, California, USA

Received 1 October 2005; accepted 10 November 2005
Communicated by C. Brezinski and R. S. Varga

In memory of Hans Zassenhaus, 1912–1991, who turned to computational mathematics after distinguished work in algebra

The triangular factorization of symmetric matrices, usually ascribed to A. L. Cholesky, was (essentially) explicitly given by O. Toeplitz earlier, but rather parenthetically. Actually, the word 'matrix' does not appear in the report on Cholesky's process; the matrix formulation seems due to Henry Jensen.

Keywords: linear systems, Cholesky method, Toeplitz, triangular factorization, symmetric matrices

$$D_0 := 0, \quad D_k = \det\left(A^{(k)}\right), \quad k = 1, 2, \ldots, n.$$

Theorem (O. Toeplitz, 1907)

If A is an $n \times n$ symmetric matrix, and $D_1, \ldots, D_n$ are all different from zero.

Then $A = LL^T$ where

$$l_{ij} = \begin{cases} \frac{1}{D_{j-1}j} \det\left(A_{1:i,\{i:i-1,j\}}\right), & i = j, j+1, \ldots n, j = 1, 2, \ldots n. \\ 0 & i < j \end{cases}$$

The aim of this has been:

- To present an account of an important but sometimes overlooked idea on linear algebra;
- To provide a carrier for some mild propaganda for numerical analysis;
- To put on record some of the important contributions to mathematics, and numerical mathematics in particular, made by John Todd.

Thank you!