MA211 : Calculus, Part 1

Lecture 2: Sets and Functions

Dr Niall Madden (Mathematics, NUI Galway)

Wednesday, 10 September 2008

Outline

Welcome to MA211

(*The next 13 frames contain a short summary of the information provided at Monday's Lecture*)

This is Semester 1 of the Second Year Calculus course.

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Schedule

Lectures: Monday and Wednesday, 11-12 in the Cairnes Lecture Theatre.

Tutorials: *Provisional Schedule*
- Tuesdays 3-4, AC202
- Wednesdays 5-6 QA003 Physiology
- Thursdays 6-7 IT205

Tutorials will being during week 3.

Website

The on-line resources for this course are on `http://BlackBoard.NUIGalway.ie`.

There you'll find various pieces of information, including these notes.

It may take a week or two for everyone to have access to BlackBoard. In the short term, we'll also use `http://www.maths.NUIGalway.ie/MA211`.

Lecture Notes

A **summary of each lecture** will be posted to the site no later than 9.30 on the day of the lecture.

Print these out and bring them with you to lectures.

You will then annotate these notes during the class.

Topics

The key topics in **MA211** are (but not in order)

1. Sets and functions.
2. Methods of integration: substitution, integration by parts, partial fractions, reduction formulae.
3. Improper integrals (as limits of finite integrals).
4. Differential equations: linear equations with constant coefficients, first order homogeneous equations, boundary value problems, etc.

Mathematical Preliminaries

Anyone who can remember their first year calculus should be able for this course.

Where we make particular use of topics from 1st year, I will try to remind you and give you a reference for a text-book.

If I don't, please ask!

Text book

There is no required textbook for this course, but two are particularly recommended:

1. Stewart: *Calculus* and *Calculus: early transcendentals*. There are copies in the library, particularly of "early transcendentals".
2. Robert Adams, *Calculus: a short course, 3rd ED*, 515 ADA There are 7 copies in the library.

If you buy a copy of either of these, you will find it useful for MA211.

Text book

Also useful are:

1. Anton, *Calculus*, 515 ANT
2. Spiegel, *Advanced Calculus*, 515 SPI (12 copies in the library) – This is only for the 1st half of the course.

In general: any book with *Calculus* in the title and that covers

- Integration, including Improper Integrals
- Transcendental functions, in particular exponential, logarithmic and hyperbolic functions.
- Differential equations.

Course assessment

Your progress in, and commitment to, this course will be assessed as follows:

- **Homework Assignment:** There will be exercises included in every lecture. These will be collected into a series a problem sets which will be posted separately to Blackboard.
  Every 3 weeks (approximately) you will be required to submit *carefully written solutions* to selected exercises. These will be graded and returned to you. The mark you get will count towards you final MA211 grade
- **Class Test:** There will be a 30 minute class test during Week 6.
- **End of Semester Exam:** Worth **75%** of the total grade for MA211.

What is Calculus?

Wikipedia: *Calculus (from Latin, "pebble" or "little stone") is a branch of mathematics that includes the study of limits, derivatives, integrals, and infinite series, and constitutes a major part of modern university education.*

Calculus has widespread applications in science and engineering and is used to solve complex and expansive problems for which algebra alone is insufficient.

It builds on analytic geometry and mathematical analysis and includes two major branches, differential calculus and integral calculus, that are related by the fundamental theorem of calculus.

Exercises

Exercise (1.1)

Go to to the library. Find where they keep the calculus books. Choose any three. Find the section where they introduce the concept of a **limit** of a function at a point. Write down the *definition of a limit* they provide, *their explanation of what it means*, and *One example*.
Rank the books in order of how useful you think they are.

Exercise (1.2)

The study of what we call "Calculus" is said to have been started by *Isaac Newton* and *Gottfried von Leibniz*.
Find out when and where they lived, and what their major mathematical discoveries were.

Short review of sets

A **set**, roughly speaking, is a collection of objects. It the set is made up of, for example, a, b and c, we write the set as

$$\{a, b, c\}$$

We give the set a name. For example, call it S, and write

$$S = \{a, b, c\}$$

Short review of sets

When talking about the set S, we might say things like

- "*a is an element of S*",
- "*a is an member of S*",
- "*S contains a*".

and we write $a \in S$.

We call the set B a *subset* of A if every element of B is also in A.

Example

Let $S = \{a, b, c\}$ and $T = \{a, c\}$, then T is a subset set of A. We write $T \subseteq S$.

Sets of numbers — The Naturals: $\mathbb{N}$

The most fundamental or "*natural*" set of numbers is, well, the **Natural Numbers:** $\mathbb{N} = \{1, 2, 3, \dots\}$. These are the ones we use for counting.

But we can't use the for solving even simple *linear* equations such as: *find x such that*

$$x + 2 = 0.$$

The solution to this is $x = -2$ which is not a natural number. So we must also use negative numbers...

The next most obvious set of numbers is the **Integers**

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$ [1]

Note that every natural number is also an integer. Mathematically, we write this as $\mathbb{N} \subset \mathbb{Z}$. In English we read this as "*the Natural numbers are contained in the set of Integers*".[2]

[1] We use the symbol $\mathbb{Z}$ because of the German word *Zahlen*, meaning Integer

[2] Or "*The Natural numbers is a subset of the Integers*". Or "*The Integers is a superset of the Naturals*".

But even the Integers is not a big enough set to contain the solutions to even very simple equations. For example: *find x such that*

$$3x - 2 = 0.$$

The solution to this is $x = 2/3$ which is not an integer. So we have to include fractions...

The next set of number we try is called the *Rationals*, denoted by the symbol $\mathbb{Q}$ [3] which is made up of all the numbers are can be expressed as fractions.

More precisely:

$$\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z},\ q \neq 0\}.$$

This means that a number is *rational* if can be written as the ratio of two integers.

Examples:

- $3\frac{3}{7} = \frac{24}{7}$ is rational.
- $2.14 = 107/50$ is rational
- $\pi = 3.14159265358979...$ is *not* rational - it has an infinite decimal expansion.

[3] $\mathbb{Q}$ stands for Quotient

A more pertinent example of numbers that are not rational are the solutions to the equation

$$x^2 - 2 = 0.$$

The solutions: $x = \sqrt{2}$ and $x = -\sqrt{2}$ are not rational.

Exercise (2.1)

Show that $\sqrt{2}$ is *not* a rational number. That is, show that there is no pair on integers a and b such that a and b have no common divisors and $(a/b)^2 = 2$.
Hint: See Proposition 2.6 in Smith's *Introductory Mathematics*.

Sets of numbers — The Rationals: $\mathbb{Q}$

Note that all the Integers (and so too the Naturals) are contained in the Rationals:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}.$$

Sets of numbers — Real Numbers: $\mathbb{R}$

The *real numbers:* $\mathbb{R}$, are *limits of sequences* of rational numbers such as

$$\sqrt{2} = 1.41421356\ldots,$$

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots.$$

$\mathbb{R}$ is set of numbers system you have used most often, and it is the one we are most concerned about in this course.

Exercise (2.2)

What sets are usually represented by the symbols

$$\mathbb{R},\ \mathbb{N},\ \mathbb{Z},\ \mathbb{Q} \text{ and } \mathbb{C}?$$

For each one, determine which of the others it is a subset of.

Sets of numbers — Real Numbers: $\mathbb{R}$

Some Notation

When we want to refer to some subset of the real numbers, we sometimes write it as

- $[a, b]$ meaning all numbers x such that $a \leq x \leq b$.
- (a, b) meaning all numbers x such that $a < x < b$.

Example

- All x strictly between 3 and 4:
- All x less that -1
- All x greater than or equal to 4

A short review of functions — Domain and Codomain

The idea of a *function* is one of the key concepts in mathematics.

Definition (Function)

Given two sets X and Y, a **function** f from X to Y is a rule of correspondences that associates every element of X with some (single) element of Y. We write:

$$f : X \to Y.$$

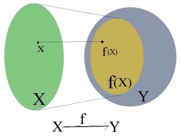
- X is called the **Domain** of f, and
- Y is called the *Codomain*.

If $x \in X$ is mapped to y in Y then we write $f(x) = y$.
Can also say "*f sends x to y*" or "*x is the image of y (under f)*".

If $f : A \to B$ the subset of elements of B that are images of elements of A is called *Range* of f, or the **Image** of f.

Definition (Range)

y is in the range of $f : A \to B$ if there is some $x \in A$ such that $f(x) = y$.



In contrast with how it was presented in the last few slides, we most often define a function simply by giving a formula for it, and leave it to the reader to decide what the domain and range are.

Example (2.1)

Find the subsets of $\mathbb{R}$ that are the largest possible domain and range for the function

$$f(x) = x^2 + 1.$$

Example (2.2)

What is the largest domain and range for the function $f(x) = \sqrt{x+2}$?

Example (2.3)

Determine the largest domain and range for the function

$$f(x) = \frac{1}{x^2 - x}.$$

Exercise (2.3)

For each of the following, give the largest possible subset of $\mathbb{R}$ that can be the domain and range:

(i) $f(t) = 1/(1+t)$
(ii) $f(x) = \sqrt{9 - x^2}$
(iii) $f(x) = \cos(x)$
(iv) $f(t) = \sin(5t - 2)$.
(v) $f(x) = 1 + \dfrac{1}{1 - x^2}$.
(vi) $f(x) = e^x$.

Even and Odd functions

Definition

A function f is called **even** if $f(-x) = f(x)$.

A function f is called **odd** if $f(-x) = -f(x)$.

It is possible for a function to be nether **odd** nor *even*.

Example

Show that the function $f(x) = x^2$ is *even*.

Even and Odd functions

Example

Show that the function $f(x) = \dfrac{x^3 - x}{x^2 + 1}$ is **odd**.

Even and Odd functions

Example

Recall that the *absolute value function* is defined as

$$\|x\| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

1. Is the function $f(x) = |x|$ odd or even?
2. Is the function $f(x) = |3 - x|$ odd or even?

Exercise (2.4)

For each of the following functions, determine if it is even, odd, or neither.

(i) $f(x) = \dfrac{x}{x^2 + 1}$

(ii) $f(x) = \dfrac{x^2}{x^4 + 1}$

(iii) $f(x) = x|x|$

(iv) $f(t) = \dfrac{t^3 + 3t}{t^4 - 3t^2 + 4}$

(v) $f(x) = 2 + x^2 + x^4$

Exercise (2.5)

Are the trigonometric functions sin, cos and tan even, odd, or neither?