MA211

Lecture 5: Calculating Derivatives

Monday 22 September 2008

ANALYSE
DES
INFINIMENT PETITS,
Pour l'intelligence des lignes courbes.

A PARIS,
DE L'IMPRIMERIE ROYALE.
M. DC. XCVI.

Blackboard

The Blackboard site is now live. I won't be updating the pages at `http://www.maths.nuigalway.ie/MA211/`

If you are registered for MA211, you should be able to access it. If for some reason you can't, then send me an email.

Problem Solving Sessions

Problem Solving Sessions start this week

- Tuesday, 3pm, AC202
- Wednesday, 5pm, QA003 (Physiology lecture room)

A problem sheet will be posted to the website later today (Monday 22/09/08).

Outline

Recall... Derivatives

Last week we began studying the *differentiation* of functions, and discovered that

1. The formal definition of the derivative (w.r.t x) of a function f is
$$f'(x) := \frac{d}{dx}f(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

2. If $f(t) = c$ (a constant), then $f'(t) = 0$.

3. If $f(t) = t$, then $f'(t) = 1$.

4. If $f(t) = t^n$, then $f'(t) = nt^{t-1}$.

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Recall... Derivatives

More Generally:

5. $\big(f+g\big)'(x) = f'(x) + g'(x)$.
6. $\big(f-g\big)'(x) = f'(x) - g'(x)$.
7. $\big(Cf\big)'(x) = Cf'(x)$, for a constant C.
8. The "Product Rule": $\dfrac{d}{dx}(u \cdot v)(x) = u(x)v'(x) + u'(x)v(x)$.
9. The "Quotient Rule" $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)(x) = \dfrac{v(x)u'(x) - u(x)v'(x)}{\big(v(x)\big)^2}$.
10. These "rules" and the derivatives of many common functions are to be found on p41 and p42 of the Mathematics Tables.

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Trigonometric function

(See, e.g., Stewart *Calculus (Early Transcendentals), Section 3.2)*

We could differentiate the trigonometrical functions cos and sin from first principles and we would find

$$\frac{d}{dt}\cos(t) = -\sin(t),$$

and

$$\frac{d}{dt}\sin(t) = \cos(t).$$

The key to these proofs is that

$$\lim_{t \to 0} \frac{\sin(t)}{t} = 0.$$

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Trigonometric function

Example

Find the derivative, with respect to x of

$$f(t) = t + t^2 \sin(t).$$

Trigonometric function

Example

Find the derivative, with respect to t of

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Trigonometric function

Example

Use the Quotient Rule to evaluate the derivative, w.r.t x of

$$f(x) = \frac{\sin(x)}{2x}.$$

Trigonometric function

Exercise (5.1)

(i) Working from 1st principles, show that

$$\frac{d}{dx}\sin(x) = \cos(x).$$

Hints:

- $\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$
- $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b).$

(ii) Use the "Quotient Rule" and the fact that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ to find the derivative of $\tan(x)$ with respect to x.

Exercise (Q5.2)

Use the product and quotient rules to evaluate the derivatives (with respect to x) of the following functions

(i) $f(x) = xe^x$,

(ii) $f(x) = \dfrac{x^3}{1 - x^2}$

(iii) $f(x) = x^2 \sin(x)$

Chain Rule

(See *Calculus*, Section 3.4)

Of all the techniques for differentiation, the most important is the *Chain Rule* for differentiating the composition of two functions.

Theorem (The Chain Rule)

Suppose that $f(x)$ *is defined as*

$$f(x) = h\big(g(x)\big).$$

Then

$$\frac{df}{dx} = \frac{dh}{dg}\frac{dg}{dx}.$$

Chain Rule

Example

Calculate the derivative of

$$f(x) = (x^2 - 1)^3.$$

Chain Rule

Example

Differentiate the function $f(x) = \sqrt{x^2 + 1}$ with respect to x

Chain Rule

Example

Find the derivative of $f(x) = e^{\sin(x)}$

Chain Rule

Exercise (Q5.3)

Use the Chain Rule to evaluate the derivative (with respect to x) of each the following functions:

(i) $f(x) = \sin(x^2)$.

(ii) $f(x) = \cos(k^2 + x^2)$.

(iii) $f(x) = \dfrac{1}{\sqrt[3]{x^2 + x + 1}}$

(iv) $f(x) = \dfrac{x}{(x^4 + 1)^3}$

(v) $f(x) = xe^{-kx}$

(Note: k is a constant independent of x)

Exercise (Q5.4)

Use the Product Rule and Chain Rule together to deduce the Quotient Rule.

l'Hospital's Rule

Recall again the problem of calculating limits of the form

$$\lim_{x \to a} \frac{f(a)}{g(a)}$$

where

- $f(a) = g(a) = 0$
- $f(a) = g(a) = \pm\infty$

Then, by l'Hospital's Rule,

$$\lim_{x \to a} \frac{f(a)}{g(a)} = \lim_{x \to a} \frac{f'(a)}{g'(a)}$$

Example

Evaluate the limit of the function

$$\frac{\log(x)}{x-1}$$

as x tends to 1.

l'Hospital's Rule

Example

Find $\lim_{x \to \infty} \frac{e^x}{x^2}$.

(Note: you can use l'Hospital's rule repeatedly.)

Example

Use *l'Hospital's Rule* to find $\lim_{x \longrightarrow 0^+} x \log(x)$

l'Hospital's Rule

Exercise (Q5.5)

Use *l'Hospital's Rule* to evaluate the following limits:

1. $\lim_{x \to 0} \dfrac{\sin(x)}{x}$
2. $\lim_{x \to 0} \dfrac{x^2 + 1}{x + 1}$
3. $\lim_{x \to 0} \dfrac{1 - \cos(x)}{x^2}$