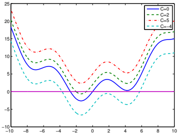
MA211
# Lecture 6: Antiderivatives and Integrals

Wed 24 September 2008






## Blackboard

From today (24/09/08) I won't be updating the pages at http://www.maths.nuigalway.ie/MA211/

If you are registered for MA211, you should be able to access all course material through http://blackboard.nuigalway.ie

If for some reason you can't, then send me an email.



## Problem Set 1

Problem Set 1 is available for down-load.
Write out, clearly and carefully, solutions to the selected exercises and submit them by 11am, Monday Oct 6th.

However, you should attempt **all** exercises. Some of them may feature on the final exam.

Tutorials take place

- Tuesday, 3pm, AC202
- Wednesday, 5pm, QA003 (Physiology lecture room)



## In today's class...

See Stewart's *Calculus* 5.3

On Monday we considered problems of the form: *given a function $f$ find it's derivative*. That is, find $g$ such that $g(x) = \frac{d}{dx}f(x)$.

However, much of this course is related to the *inverse* of this problem: *given a function $f$ find it's* **antiderivative**

**Definition (Antiderivative)**

Given a function $f$ in an interval $I$, the function $F$ is *an* **antiderivative** of $f$ on $I$ if

$$F'(t) = f(t) \quad \text{for all } x \in I.$$





**Example**

- $F(t) = t$ is an antiderivative of $f(x) = 1$.
- $F(t) = \frac{1}{2}t^2$ is an antiderivative of $f(t) = t$.
- $F(t) = -cos(t)$ is an antiderivative of $f(t) = \sin(t)$.





Note that $F(t) = -1/t$ is an antiderivative of $f(t) = 1/t^2$ (*on any interval that excludes $t = 0$*).

But so too is $F(t) = 5 - 1/t$ and $F(t) = -1/t - 3.1415$ and, indeed, any function of the form $F(t) = -1/t + C$ for some constant $t$.

When we write down the antiderivative of $f$ and include the constant $C$ we usually call it the **General Antiderivative** of $f$ or, more commonly, the *The Indefinate Integral*.





**Definition (Indefinite Integral)**

The **Indefinite Integral** of $f(t)$ on the interval $I$ is

$$\int f(t)dt = F(t) + C \quad \text{for } t \in I,$$

where $F'(t) = f(t)$ for all $t$ in $I$.

We call $C$ the *constant of integration*.

(*Next week we'll do* **definate integrals**, *which have limits of integration:* $\int_a^b f(x)dx$).

1. $\int 1 \; dx =$

2. $\int x \; dx =$

3. $\int x^2 dx =$

4. $\int x^n dx =$





5. $\int \sin(x) dx =$

6. $\int \cos(x) dx =$



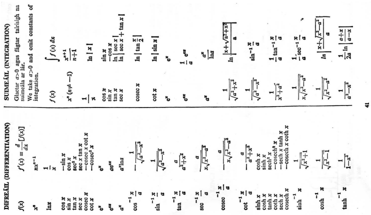


It is neither important or necessary to memorise the antiderivatives of even reasonably common functions. However, you should be able to look them up on pages 41 and 42 of the Mathematical Tables.

Having p9 is also handy.







**DIFREÁIL (DIFFERENTIATION)**

| $f(x)$ | $f'(x) \equiv \frac{d}{dx}[f(x)]$ |
|---|---|
| $x^n$ | $nx^{n-1}$ |
| $\ln x$ | $\frac{1}{x}$ |
| $\cos x$ | $-\sin x$ |
| $\sin x$ | $\cos x$ |
| $\tan x$ | $\sec^2 x$ |
| $\sec x$ | $\sec x \tan x$ |
| $\operatorname{cosec} x$ | $-\operatorname{cosec} x \cot x$ |
| $\cot x$ | $-\operatorname{cosec}^2 x$ |
| $e^x$ | $e^x$ |
| $e^{ax}$ | $ae^{ax}$ |
| $a^x$ | $a^x \ln a$ |
| $\cos^{-1}\frac{x}{a}$ | $-\frac{1}{\sqrt{a^2-x^2}}$ |
| $\sin^{-1}\frac{x}{a}$ | $\frac{1}{\sqrt{a^2-x^2}}$ |
| $\tan^{-1}\frac{x}{a}$ | $\frac{a}{a^2+x^2}$ |
| $\sec^{-1}\frac{x}{a}$ | $\frac{a}{x\sqrt{x^2-a^2}}$ |
| $\operatorname{cosec}^{-1}\frac{x}{a}$ | $-\frac{a}{x\sqrt{x^2-a^2}}$ |
| $\cot^{-1}\frac{x}{a}$ | $-\frac{a}{a^2+x^2}$ |
| $\sinh x$ | $\cosh x$ |
| $\cosh x$ | $\sinh x$ |
| $\tanh x$ | $\operatorname{sech}^2 x$ |
| $\coth x$ | $-\operatorname{cosech}^2 x$ |
| $\operatorname{sech} x$ | $-\operatorname{sech} x \tanh x$ |
| $\operatorname{cosech} x$ | $-\operatorname{cosech} x \coth x$ |
| $\sinh^{-1} x$ | $\frac{1}{\sqrt{x^2+1}}$ |
| $\cosh^{-1} x$ | $\frac{1}{\sqrt{x^2-1}}$ |
| $\tanh^{-1} x$ | $\frac{1}{1-x^2}$ |

**SUIMEÁIL (INTEGRATION)**

Glactar $a>0$ agus fágtar tairisigh na suimeála ar lár.
We take $a>0$ and omit constants of integration.

| $f(x)$ | $\int f(x)\,dx$ |
|---|---|
| $x^n \; (n \neq -1)$ | $\frac{x^{n+1}}{n+1}$ |
| $\frac{1}{x}$ | $\ln\lvert x\rvert$ |
| $\cos x$ | $\sin x$ |
| $\sin x$ | $-\cos x$ |
| $\tan x$ | $\ln\lvert\sec x\rvert$ |
| $\sec x$ | $\ln\lvert\sec x + \tan x\rvert$ |
| $\operatorname{cosec} x$ | $\ln\lvert\tan\frac{x}{2}\rvert$ |
| $\cot x$ | $\ln\lvert\sin x\rvert$ |
| $e^x$ | $e^x$ |
| $e^{ax}$ | $\frac{1}{a}e^{ax}$ |
| $a^x$ | $\frac{a^x}{\ln a}$ |
| $\frac{1}{\sqrt{a^2+x^2}}$ | $\ln\left\lvert\frac{x+\sqrt{a^2+x^2}}{a}\right\rvert$ |
| $\frac{1}{\sqrt{a^2-x^2}}$ | $\sin^{-1}\frac{x}{a}$ |
| $\frac{1}{x^2+a^2}$ | $\frac{1}{a}\tan^{-1}\frac{x}{a}$ |
| $\frac{1}{x\sqrt{x^2-a^2}}$ | $\frac{1}{a}\sec^{-1}\frac{x}{a}$ |
| $\frac{1}{\sqrt{x^2-a^2}}$ | $\ln\left\lvert\frac{x+\sqrt{x^2-a^2}}{a}\right\rvert$ |
| $\frac{1}{a^2-x^2}$ | $\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert$ |

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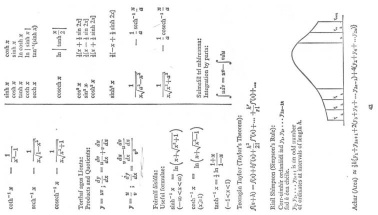
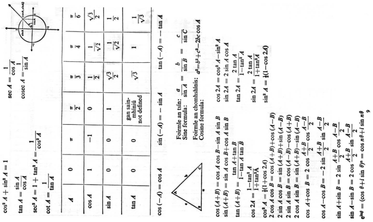












**Example**

1. $\int (x^3 - 5x^2 + 7)\, dx =$

2. $\int x^{-3} dx =$





**Example**

3. $\int x^{-n} dx =$

4. $\int x^{-1} dx =$

5. $\int \frac{\sin(t)}{\tan(t)} dt =$

Don't forget the constant of integration!

**Exercise (Q6.1)**

(i) $6t^2 - 1$,

(ii) $\dfrac{x+3}{x^{3/2}}$

(iii) $\int 6dx$

(iv) $\displaystyle\int x^{-2}dx$

(v) $\displaystyle\int \left(x^2 + \cos(x)\right)dx$

(vi) $\int \cos(t)\tan(t)dt$

(vii) $\displaystyle\int (A + Bx + Cx^2)dx$

(viii) $\displaystyle\int \cos(3x)dx$

Don't forget the constant of integration!



## Differential equations

When we see a problem like:

Evaluate $\displaystyle\int 3t^2 - 1dt$

we can think of it as

Find a function whose derivative (with respect to $t$) is $3t^2 - 1$.

Another equivalent way of asking the same question is:

Find a function $f$ that solves the equation $f'(t) = 3t^2 - 1$.

This is an example of a simple *Differential Equation (DE)*, and we'll study much more of these as go through the course.



## Differential equations

Our 1st Differential Equation is:

**Example (1)**

Find a function $f$ that solves the equation $f'(t) = 3t^2 - 1$.

and its solution is of the form

$$f(t) = t^3 - t + C$$

for an *arbitrary* constant $C$.

**Definition (General Solution)**

The **general solution** of a differential equation is one that includes one or more arbitrary constants corresponding to constants of integration.



## Differential equations

**Example (2)**

Find the general solution to the differential equation

$$f''(x) = x.$$

## Differential equations

**Example (3)**

Show that the function

$$f(x) = C_1 x^3 + C_2/x$$

is a solution to the differential equation

$$x^2 f''(x) - x f'(x) - 3f(x) = 0.$$



## Differential equations

**Exercise (Q6.2)**

(i) Show that, for any constants $C_1$ and $C_2$,

$$y(x) = C_1 x^2 + C_2 x^{-2}$$

is a solution to the differential equation

$$x^3 y'''(x) + 6x^2 y''(x) = 12y(x).$$

(ii) Write down a 2nd order differential equation that has $f(x) = x^2 - x$ as a solution.



## Differential equations

**Example (4)**

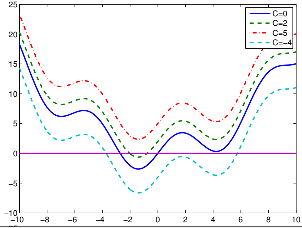
Write down the **general solution** to the following differential equation:

$$y'(x) = x/3 + 3\cos(x)$$



## Differential equations

$$\frac{x^2}{6} + 3\sin(x) + C$$



## General V Particular Solutions

The following is an example of a simple differential equation:

**Q:** If you travel east at a constant speed of 90km/hr for 1 hour, where are you?
*A:* 90km east of where we started!
This is the *general solution*.

An alternative problem is:

**Q:** If you travel east *from Galway* at a constant speed of 90km/hr for 1 hour, where are you?
*A:* Athlone.
This is a *particular solution*: the arbitrary constant is specified.



## Particular Solutions

**Example (5)**

Find the solution to the differential equation

$$y'(x) = \frac{x}{3} + 3\cos(x),$$

given that $y(0) = 2$.



## Particular Solutions

**Exercise (Q6.3)**

Find solutions to the following differential equations. If possible, gave a particular solution, otherwise, give the general solution.

(i) $y'(t) = x - 2$
(ii) $f'(x) = x^{-2} - x^{-3}$, subject to $f(-1) = 0$.
(iii) $y''(x) = x^3 - 1$, given that $y'(0) = 0, y(0) = 8$.