MA211

Lecture 9: 2nd order differential eqns

Monday, $6^{\rm th}$ October 2008

Class test next week...

This morning

1. Recall... The Hyperbolic Functions
   - Properties
   - Examples
2. More about Hyperbolic Functions
3. Differential Equations
4. Linear Combinations of Solutions
5. The Axillary Equation
6. $D > 0$

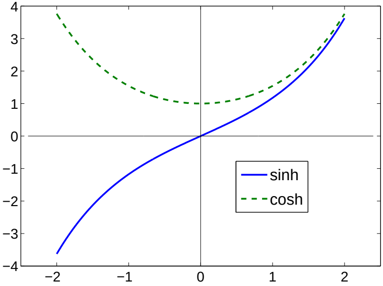
For more details, see **17.1** of Stewart.

Recall... The Hyperbolic Functions

Definition (Hyperbolic Functions)

The **Hyperbolic cosine and sine functions** are defined as

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x}), \quad \sinh(x) = \frac{1}{2}(e^x - e^{-x})$$



Last week we saw that:

- $\dfrac{d}{dx}(\sinh x) = \cosh x$
- $\dfrac{d}{dx}(\cosh x) = \sinh x$

Example

Show that $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

(To show this is true, we repeatedly use that $e^a e^b = e^{a+b}$).

Example (Q1 (b), Semester 1, 05/06 (v))

Prove that

$$\frac{d}{dx}\left(\cosh^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{x^2 - a^2}}.$$

Hint: use the Chain Rule and that $\cosh^2 y - \sinh^2 y = 1$.

Exercise (Q9.1)

(i) Recall that $\cos^2 x + \sin^2 x = 1$. Show that $\cosh^2 x - \sinh^2 x = 1$.

(ii) What are the largest possible domain for the functions $f(x) = \sinh(x)$ and $f(x) = \sinh^{-1}(x)$? Sketch their graphs.

(iii) Show that $\sinh(2x) = 2\cosh(x)\sinh(x)$

(iv) Prove that

$$\frac{d}{dx}\left(\sinh^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 + x^2}}.$$

(v) Show that

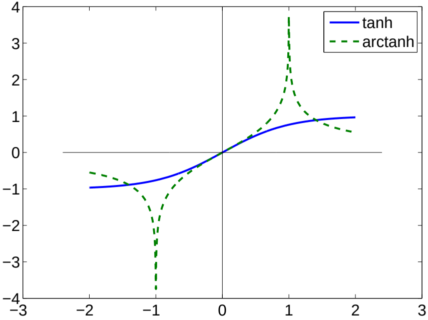
$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y.$$

At least one of these will appear on next Wednesday's class test.

More about Hyperbolic Functions

The tanh and *cotanh* functions can be defined

$$\tanh x = \frac{\sinh x}{\cosh x}, \qquad \coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}.$$



More about Hyperbolic Functions

Exercise (Q9.2)

Show that

(i) $\tanh(x) = \frac{e^{2x}-1}{e^{2x}+1}$

(ii) $\cosh^{-1}(x) = \ln(x + \sqrt{x^2+1})$

(iii) $\dfrac{d}{dx}\tanh(x) = 1 - \tanh^2(x)$

(iv) $\dfrac{d}{dx}\tanh^{-1}\left(\dfrac{x}{a}\right) = \dfrac{1}{a^2 - x^2}$

(v) $\cosh(2x) = \cosh^2(x) + \sinh^2(x)$

(vi) $\cosh(x) + \sinh(x) = e^x$

(vii) $\cosh(x) - \sinh(x) = e^{-x}$

Now that we have the exponential, logarithmic, trigonometric and hyperbolic functions at our disposal, we can solve some differential equations.

The DEs that we'll look at now are of

2nd Order, Constant Coefficient, Homogeneous type.

Example

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0 \iff y''(x) + y'(x) - 2y(x) = 0.$$

2nd Order, Constant Coefficient, Homogeneous type.

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \iff ay''(x) + by'(x) + cy(x) = 0.$$

where a, b and c are constants (real numbers).

We will see that all the solutions to these equations come in one of the following forms:

1. $Ae^{R_1 x} + Be^{R_2 x}$,
2. $(Ae + Bx)e^{Rx}$.
3. $e^{kx}\big(A\cos(\omega t) + B\sin(\omega t)\big)$

where A and B are arbitrary constants.

Differential Equations

To solve these equations we will:

- First assume that the solution is $y = Ce^{Rx}$.
- Substitute this into the DE to get a quadratic equation for R.
- Call the two solutions to this equation R_1 and R_2.
- Where its useful, we'll express the solutions in terms of trig functions using **Euler's Formula**:

$$e^{ix} = \cos(x) + i\sin(x) \qquad \text{where } i = \sqrt{-1}$$

Linear Combinations of Solutions

Suppose that y is a solution to the differential equation

$$ay'' + by' + cy = 0,$$

Then so too is Ky for any constant K

Linear Combinations of Solutions

If y_1 and y_2 are both solutions to

$$ay''(x) + by'(x) + cy(x) = 0,$$

The so too is any function $y(x) = Ay_1(x) + By_2(x)$.

Linear Combinations of Solutions

Example

Find r such that $y(x) = e^{Rx}$ is a solution to the equation:

$$y'' + 5y' + 4y = 0.$$

The Axillary Equation

In the previous example, the key part is solving the quadratic equation

$$aR^2 + bR + c = 0.$$

$$R = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This is called **The Auxiliary Equation**.

Also, $D = b^2 - 4ac$ is called the **Discriminant**.

- If $D > 0$, then there are two real-valued solutions to the auxiliary equation.
- If $D = 0$, then the auxiliary equation has only one solution.
- If $D < 0$, the solutions to the auxiliary equation are *complex valued*.

$D > 0$

The easiest case is $D = b^2 - 4ac > 0$.

$D > 0$

If $D = b^2 - 4ac > 0$, then the auxiliary equation

$$ar^2 + br + c = 0$$

has two solutions:

$$R_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad R_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

and the general solution is

$$y(x) = Ae^{R_1 x} + Be^{R_2 x}.$$

$D > 0$

Example

Write down the general solution to the differential equation

$$y'' - 2y' - 3y = 0.$$

Verify your answer is correct.

Solution: