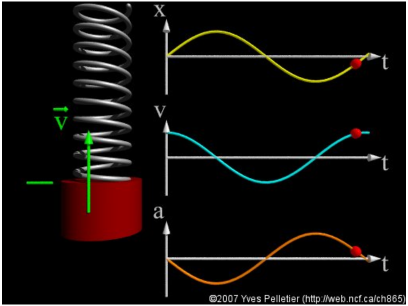
MA211

Lecture 10: 2^{nd}-Order DEs with Constant Coefficients

Wednesday, 8^{th} October 2008



Class test next Wednesday

Reminder: There will be a 30 minute in-class test next Wednesday (15/10/08). It will be worth approximately 5% for total for MA211.

Questions will be based on **Problem Set 2**.

In this class...

1. Recall...
   - $D > 0$
2. $D = 0$
3. $D < 0$
   - Simple Harmonic Motion

For more details, see **17.1** of Stewart.

Recall...

2nd Order, Constant Coefficient, Homogeneous differential equations

On Monday we started a new section of MA211 were we try to solve problems of the form

$$ay''(x) + by'(x) + cy(x) = 0.$$

where a, b and c are constants (real numbers).

We introduced the *The Auxiliary Equation*:

$$aR^2 + bR + c = 0,$$

and the **Discriminant**, $D = b^2 - 4ac$.

Recall...

$$ay''(x) + by'(x) + cy(x) = 0.$$

where a, b and c are constants (real numbers).

When solving the above equation, we consider separately the three cases

(i) $D > 0$, (ii) $D = 0$ (iii) $D < 0$.

The easiest case is $D = b^2 - 4ac > 0$.

$D > 0$

If $D = b^2 - 4ac > 0$, then the auxiliary equation

$$ar^2 + br + c = 0$$

has two solutions:

$$R_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad R_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

and the general solution is

$$y(x) = Ae^{R_1 x} + Be^{R_2 x}.$$

Example

Find the general solution to the differential equation

$$y'' - 4y = 0.$$

and express the solution in terms of sinh and cosh

$$\cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right), \quad \sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$$

Solution:

Exercise (Q10.1)

Find general solutions to the following differential equations:

(i) $y'' + y' - 6y = 0$.

(ii) $3y'' + y' - y = 0$.

(iii) $y'' + 4y' + 2y = 0$

(iv) $y'' + 2y' = 0$

$D = 0$

The next easiest case is $D = b^2 - 4ac = 0$.

$D = 0$

If $D = b^2 - 4ac = 0$, then the auxiliary equation

$$ar^2 + br + c = 0$$

has just one solution:

$$R = \frac{-b}{2a},$$

and the general solution is

$$y(x) = Ae^{Rx} + Bxe^{Rx}.$$

$D = 0$

Example

Find the general solution to the equation

$$y'' + 2y' + y = 0,$$

and verify your solution.

Solution:

$D = 0$

Example

Find the general solution to the equation

$$4y'' + 12y' + 9y = 0.$$

$D = 0$

Example

Suppose the coefficients of the differential equation

$$ay'' + by' + cy = 0.$$

are such that $b^2 = 4ac$. If $y_1 = e^{Rx}$ is a solution, where $R = -b/2a$, then show that $y_2 = xe^{Rx}$ is also a solution.

Exercise (Q10.2)

Find general solutions to the following differential equations:

(i) $\frac{3}{4}y'' + 3y' + 3y = 0$.

(ii) $y'' - 8y' + 16y = 0$.

$D < 0$

Finally, we consider the most complicated situation:

$D < 0$

$$D = b^2 - 4ac < 0,$$

so that the solutions to the auxiliary equation are *complex valued*.

But first... *simple harmonic motion*

Before we see how to solve the problem in general, we'll look at a simple but important example:

$$y'' + \omega^2 y = 0.$$

Solution: