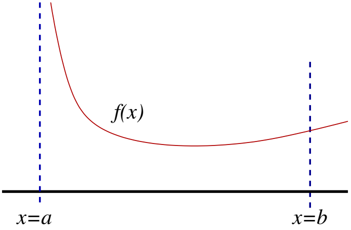
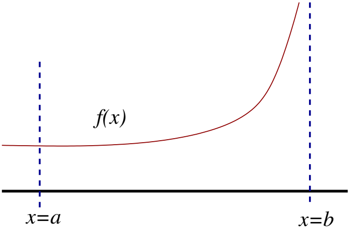
MA211

Lecture 20: Improper Integrals – Type 2

Monday 17$^{\text{th}}$ Nov 2008





Topics of the day...

1. Improper Integrals: Type 2

2. The Comparison Test

See also Section 7.7 of Stewart.

Improper Integrals: Type 2

Last week we saw how to evaluate improper integrals of *Type 1* where the limits of integration include one or both of $-\infty$ or ∞, e.g.,

Improper Integrals: Type 1

$$\int_{-\infty}^{b} f(x)dx, \qquad \int_{a}^{\infty} f(x)dx, \qquad \int_{-\infty}^{\infty} f(x)dx$$

How we'll look at Improper Integrals *of Type 2*

$$\int_{a}^{b} f(x)dx, \qquad \text{where } f(x) \to \pm\infty$$

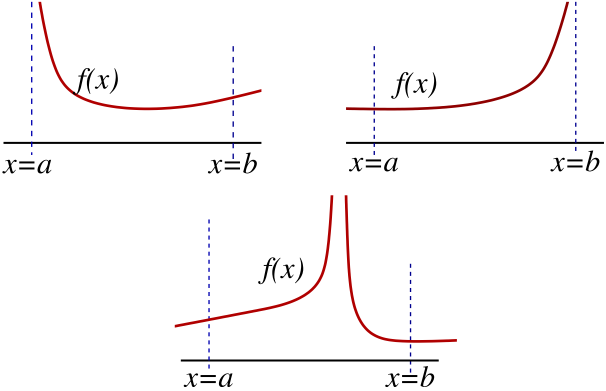
at a, b or somewhere in between.

Improper Integrals: Type 2

In particular, we want to evaluate

$$\int_a^b f(x)\,dx$$

where $f(x)$ may be unbounded at a or b, or at some point in between.



$f(x)$ unbounded at $x = a$

When function $f(x)$ is defined for $a < x \leq b$ then evaluate $\mathcal{I}(t) = \int_t^b f(x)dx$ and then use that:

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx.$$

So:

1. Evaluate $\mathcal{I}(t) = \int_t^b f(x)dx$
2. Compute the limit $L = \lim_{t \to a^+} \mathcal{I}(t)$
3. If L is finite then $\int_a^b f(x)\,dx = L$, and we can say that $\int_a^b f(x)\,dx$ **converges to** L.
4. If L is *not* finite, then integral is said to diverge.

Improper Integrals: Type 2

Example

Does the integral $\int_0^1 \frac{1}{x} dx$ converge?

Improper Integrals: Type 2

Example

Evaluate the improper integral $\int_0^1 \frac{1}{x^2} dx$

Improper Integrals: Type 2

Example

Evaluate the TYPE 2 Improper Integral $\int_0^1 \frac{1}{\sqrt{x}}dx$

Improper Integrals: Type 2

$\int_0^1 x^{-p} dx$ will *converge* when $p < 1$, and **diverge** for $p \geq 1$.

Proof: If $p = 1$ then

$$\int_t^1 x^{-p} dx = \int_t^1 \frac{1}{x} dx = \ln(x)\Big|_t^1 = \ln(t) - \ln(1) = \ln(t).$$

But $\lim_{t\to 0} \ln(t)$ does not exists, so $\int_0^1 \frac{1}{x} dx$ diverges.

If $p \neq 1$ then $\int_t^1 x^{-p} dx = \frac{x^{1-p}}{1-p}\Big|_t^1 = \frac{1 - t^{1-p}}{1-p}$.

If $p < 1$ then $1 - p > 0$ so the limit $\lim_{t\to 0} t^{1-p} = 0$. So the integral converges to $\frac{1}{1-p}$.

If however $p > 1$ then $1 - p < 0$ and $\lim_{t\to 0} t^{1-p}$ does not exist, so the integral **diverges**.

Improper Integrals: Type 2

If f is defined on $[a, b)$ and $\lim_{t \to b^-} \int_a^t f(x)\,dx$ exists, call the limit L and write

$$\int_a^b f(x)\,dx = L.$$

Again, $\int_a^b f(x)\,dx$ is said to **converge to** L. If no such limit exists, the integral is divergent.

Improper Integrals: Type 2

Example

Does the $\displaystyle\int_0^4 \frac{dx}{\sqrt{4-x}}$ converge or diverge?

Improper Integrals: Type 2

If a function f is defined on $[a, b]$ except at some point c in (a, b) at which f is *unbounded*, then use that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

The integral converges if and only if $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ **both** converge.

Example

Does the improper integral $\int_{-1}^{1} \frac{dx}{x}$ converge or diverge?

The Comparison Test

Earlier we saw how to evaluate $\int_1^\infty \frac{1}{1+x^2}\,dx$.

But suppose we just wanted to determine if it converges or diverges...

The Comparison Test

Often, we just want to know if some integral converges or diverges – and not necessarily evaluate the integral.

In that case we can compare the integral with one that we know. This is helpful because we can use the *Comparison Test*...

The Comparison Test

Comparison Test

Suppose f and g are defined on $[a, \infty)$ and

$$0 \leq f(x) \leq g(x) \text{ for all } x \in [a, \infty).$$

Then

$$\int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx.$$

Therefore

1. If $\int_a^\infty g(x)\,dx$ **converges**, so does $\int_a^\infty f(x)\,dx$
2. if $\int_a^\infty f(x)\,dx$ *diverges*, so does $\int_a^\infty g(x)\,dx$

There are corresponding results for the other types of improper integrals.

The Comparison Test

Example

Does the integral $\int_1^\infty \frac{dx}{x^2 + x^3}$ converge or diverge?

The Comparison Test

Example

Does the improper integral $\displaystyle\int_0^1 \frac{dx}{2x^2 + 3x^3}$ converge or diverge?

The Comparison Test

Example

Establish if $\displaystyle\int_0^1 \frac{dx}{2\sqrt{x}+x^2}$ is convergent or divergent.

NOTE: The solution given to this one in class was wrong.

Correct answer: For $0 \le x \le 1$ we know that $\sqrt{x} \ge x^2$, so

$$2\sqrt{x} + x^2 \ge 3\sqrt{x}.$$

Thus

$$\frac{1}{2\sqrt{x}+x^2} \le \frac{1}{3}\frac{1}{\sqrt{x}} \le \frac{1}{\sqrt{x}}.$$

But we know that $\displaystyle\int_0^1 x^{-1/2}dx$ converges, so by the Comparison Principal, so too does $\displaystyle\int_0^1 \frac{dx}{2\sqrt{x}+x^2}$.

The Comparison Test

Example

Test for convergence of the following integral:

$$\int_1^{\infty} \frac{\cos x \, dx}{1 + x^2}$$

Exercise (Q20.1)

For each of the following integrals, determine if they *converge* or *diverge*

(i) $\displaystyle\int_1^\infty \frac{|\cos(x)|}{x^3+2}\,dx.$

(ii) $\displaystyle\int_0^1 \frac{dx}{x^{5/3}}\,dx.$

(iii) $\displaystyle\int_0^1 \frac{dx}{x^{3/5}}\,dx.$

(iv) $\displaystyle\int_0^\infty \frac{x}{x^{3/2}+2x^2}\,dx.$

(v) $\displaystyle\int_{-2}^2 \frac{1}{\mathbf{x^2}}\,dx$

(vi) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x+x^4}}\,dx$