MA211

Lecture 21: 1st Order Differential Equations (Part I)

Wednesday 19$^{\mathrm{th}}$ Nov 2008

Topics of the day...

See also Sections 9.3 and 9.5 of Stewart.

Recall: 1st Order Differential Equations

At the beginning of the course, we looked briefly at 1st Order Differential Equations.

Later we studied some 2nd order problems with constant coefficients.

Now we return to 1st Order problems with non-constant coefficients. These can't be solved with the simple substitution $y = e^{rx}$: we'll need to use the Techniques of Integration that we learnt over that last 2 weeks.

Recall: 1st Order Differential Equations

As *1st Order Differential Equation* is of the form

$$\frac{dy}{dx} = f(x, y).$$

That is: the right hand side is some function of both x and y.
Solving the equation means expressing the relationship between x and y in a manner not involving derivatives.

The approaches that we will study are

1. Separable DEs (today)
2. Homogeneous DEs
3. Integrating Factors.

Separable Equations

A first order equation $\dfrac{dy}{dx} = f(x, y)$ is **separable** if we can write $f(x)$ as the product of some functions $g(x)$ and $h(y)$. That is, it has the form

$$\frac{dy}{dx} = g(x)h(y).$$

Such an equation can be solved by writing

$$\frac{1}{h(y)}dy = g(x)\,dx$$

and integrating both sides:

$$\int \frac{1}{h(y)}dy = \int g(x)\,dx$$

Separable Equations

Example

Solve the equation $\dfrac{dy}{dx} = e^{x-y}$.

Separable Equations

Example

Solve the differential equation $\dfrac{dy}{dx} = \dfrac{6x^2}{2y + cos(y)}$.

Example

Solve $\dfrac{dy}{dx} = \sqrt{y}\cos^2\left(\sqrt{y}\right)$.

Separable Equations

Example

Solve the equation $\dfrac{dy}{dx} = \dfrac{xy - 2y + x - 2}{x+3}$.

Solution: This equation is separable :

$$\frac{dy}{dx} = \frac{(x-2)(y+1)}{x+3} \Longrightarrow \frac{dy}{y+1} = \frac{x-2}{x+3}\,dx.$$

Then...

Separable Equations

Example

Solve the differential equation

$$\frac{dy}{dx} = -(y^2 + 2y + 2)(x+1)\ln(x) \text{ with } y(1) = -1$$

Separable Equations

Exercise (21.1)

Find the general solution to the 1st order separable DEs

(i) $\dfrac{dy}{dx} = \dfrac{x}{y}$

(ii) $\dfrac{dy}{dx} = \dfrac{y}{x}$

(iii) $\dfrac{\mathrm{d}\,y}{\mathrm{d}\,x} = y\ln(x)$

(iv) $\dfrac{dy}{dx} = \dfrac{y}{2x}$.

(v) $\dfrac{dy}{dx} = \dfrac{e^x}{\sin(y)}$

(vi) $\dfrac{dy}{dx} = e^y \sin(x)$.

(vii) $\dfrac{\mathrm{d}\,y}{\mathrm{d}\,x} = \dfrac{\ln(x)}{xy^2}$

Separable Equations

Exercise (21.2)

Solve the following initial value problems:

1. $\dfrac{dy}{dx} = 3 + e^{y}; \quad y(0) = 1.$
2. $\dfrac{dy}{dx} = \sinh(x)e^{-y} \quad y(0) = 1;$

Homogeneous Equations

Lectures 11 and 12 were concerned with solving *homogeneous* 2nd order differential equations with constant coefficient of the form:

$$ay'' + by' + cy = 0.$$

Here the term *homogeneous* means that the right-hand side of the equation is 0.

For first-order problems, we have a different meaning of homogeneous: if the right-hand side is a function of both x and y, but can be written in terms of a single function $v = y/x$.

Definition

A function $f(x, y)$ is **homogeneous** *of degree* k if for every real number t we have

$$f(tx, ty) = t^k f(x, y).$$

To work out if a function is homogeneous

1. Write down $f(tx, ty)$
2. Rearranged to get $t^k f(x, y)$ for some number k
3. This tells you that the function is homogeneous *and* its degree.

Important: When the function is *homogeneous of degree* 0 we can make the substitution $v = \frac{y}{x}$ and rearrange to get a function just in terms of v.

Example

For each of the following, determine if they are homogeneous. If they are homogeneous, then to what degree?

1. $f = x^2 + xy + y^2$
2. $f = \dfrac{xy + y^2}{x^2}$
3. $f = x^2 + y$
4. $f = \dfrac{y}{xy}$.

Exercise (21.3)

For each of the following, determine if they are homogeneous and to what degree.

1. $f = \sqrt{x^2 + y^2}$
2. $f = \dfrac{x^2 + xy}{xy + y^2}$
3. $f = xy^2$
4. $f = \dfrac{2xy}{x^2 + y^2}$
5. $f = \dfrac{y}{x - y}$.