MA211

Lecture 22: 1st Order Differential Equations (Part II)

Monday 24^{th} Nov 2008

Topics of the day...

1. Recall: 1st Order Differential Equations
2. Recall: Separable Equations
3. Recall: Homogeneous Functions
4. Solving homogeneous DEs
5. First Order Linear Differential Equations
6. Integrating Factors
   - Technique
   - Examples

See also Sections 9.3 and 9.5 of Stewart.

Recall: 1st Order Differential Equations

Last Wednesday, we started a new section on solving 1st order differential equations.

$$y'(x) = f(x, y).$$

Recall: Separable Equations

A first order equation $\dfrac{dy}{dx} = f(x, y)$ is **separable** if we can write $f(x)$ as the product of some functions $g(x)$ and $h(y)$. That is, it has the form

$$\frac{dy}{dx} = g(x)h(y).$$

Such an equation can be solved by writing

$$\frac{1}{h(y)}dy = g(x)\,dx$$

and integrating both sides:

$$\int \frac{1}{h(y)}dy = \int g(x)\,dx$$

Recall: Homogeneous Functions

Last Wednesday we also saw that a function $f(x, y)$ is **homogeneous** *of degree* k if for every real number t we have

$$f(tx, ty) = t^k f(x, y).$$

In interest to us is if right-hand side of the differential equation $y'(x) = f(x, y)$ is *homogeneous of degree* 0.

Then we can make the equation *separable* with the substitution $v = \frac{y}{x}$.

Solving homogeneous DEs

Given a first order differential equation $\frac{dy}{dx} = f(x, y)$ where $f(x, y)$ is *homogeneous of degree 0*,

1. Let $v = \frac{y}{x}$ and find the function h such that $h(v) = f(x, y)$.
2. Because we have $y = vx$, differentiate to get: $\frac{dy}{dx} = v + x\frac{dv}{dx}$
3. Substitute into the original differential equation:
   $$v + x\frac{dv}{dx} = h(v).$$
4. This equation involving v and x is separable:
   $$\frac{1}{h(v) - v} dv = \frac{1}{x} dx$$
5. Solve it in the same way we solve the separable problems from last week.

Solving homogeneous DEs

Example

Solve the equation $\dfrac{dy}{dx} = \dfrac{xy + y^2}{x^2}$.

Solving homogeneous DEs

Example

Solve the following initial value problem:

$$\frac{dy}{dx} = \frac{x^2 + xy}{xy + y^2}, \qquad y(2) = 1.$$

Solving homogeneous DEs

Example (Autumn exam 07/08, Q4(iii))

Solve the following differential equation:

$$2xy\frac{dy}{dx} = x^2 + y^2, \qquad y(2) = 2.$$

Solving homogeneous DEs

Exercise (22.1)

Find the general solution to the following differential equations:

1. $$\frac{dy}{dx} = \frac{x+y}{x-y}.$$
2. $$\frac{dy}{dx} = \frac{xy}{x^2+2y^2}.$$

Exercise (22.2)

Solve the following initial value problems:

1. $$\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}; \quad y(1) = 1.$$
2. $$\frac{dy}{dx} = \frac{x^3+3xy^2}{3x^2y+y^3}; \qquad y(1) = -1$$

First Order Linear Differential Equations

To finish the course, we'll look at ways of solving *first order linear* differential equation such as:

$$\frac{dy}{dx} + P(x)y = Q(x).$$

This is called *Linear* because the only expression for y is linear.

First Order Linear Differential Equations

Example

These are linear equations:

- $\dfrac{dy}{dx} - 3y = e^x$.
- $x\dfrac{dy}{dx} + y = \sin(x)$.
- $\dfrac{dy}{dx} + \sqrt{x}y = \ln(x)$.

These are **not** linear:

- $\dfrac{dy}{dx} - y^3 = e^x$.
- $y\dfrac{dy}{dx} = \sin(x)$.
- $\dfrac{dy}{dx} + x\sqrt{y} = \ln(x)$.

First Order Linear Differential Equations

A general strategy for solving such an equation is to multiply the equation by some expression $v(x)$ that simplifies the problem. Then we get:

$$v(x)\frac{dy}{dx} + v(x)P(x)y = v(x)Q(x).$$

The idea is to choose $v(x)$ so that the left hand side of the above equation is the derivative of the product vy. This would require

$$v\frac{dy}{dx} + vP(x)y = v\frac{dy}{dx} + y\frac{dv}{dx},$$

that is

$$vP(x)y = y\frac{dv}{dx} \implies \frac{dv}{dx} = vP(x).$$

Integrating Factors

So we need to choose v so that $\frac{dv}{dx} = vP(x)$. This means:

The expression $v(x)$ is called an *integrating factor* for the differential equation.

Integrating Factors

Example

Solve the differential equation $y' - 3y = e^x$.

Summary of Technique of Integrating Factors

Given a problem of the form:

$$\frac{dy}{dx} + P(x)y = Q(x).$$

1. Let $v = e^{\int P(x)dx}$.
2. Solve $(vy)' = vQ(x)$ by integrating:

$$vy = \int vQ(x)dx.$$

   not forgetting the constant of integration.
3. Divide by v to get the solution:

$$y = \frac{\int vQ(x)dx}{v}.$$

Example

Solve the equation

$$x\frac{dy}{dx} + y = \sin(x)$$

subject to the initial condition $y(\pi/2) = 1$.

Example

Solve the equation

$$x\frac{dy}{dx} - y = x^3, \quad y(1) = 1.$$

Example (Q3(c), Semester 1, '06/'07)

$$e^x \frac{dy}{dx} + 2e^x y = 1.$$

Example

Solve the following differential equation:

$$\frac{dy}{dx} + \cos(x)y = 2xe^{-\sin(x)}.$$

Exercise

Solve the following differential equations:

(i) $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2 - \dfrac{1}{x}, \qquad y(1) = 1/4.$

(ii) $\dfrac{\mathrm{d}\, y}{\mathrm{d}\, x} + 2y = e^{-x}.$

(iii) $y' = x^2 + x^2 y$

(iv) $y' + 3xy = x$

(v) $\dfrac{\mathrm{d}\, y}{\mathrm{d}\, x} = \sin(x)y = 3\sin(2x)$

(vi) $xy' + y = 2x\sin(x)$

(vii) $2xyy' = x^2 + 3y^2$

(viii) $\dfrac{\mathrm{d}\, y}{\mathrm{d}\, x} + \dfrac{y}{\tan(x)} = 3x + 1$

See also: Problem Set 5.