MA211

Lecture 23: Integrating factors and course review

Monday 24$^{\text{th}}$ Nov 2008

Summary of Technique of Integrating Factors

Given a problem of the form:

$$\frac{dy}{dx} + P(x)y = Q(x).$$

1. Let $v = e^{\int P(x)dx}$.
2. Solve $(vy)' = vQ(x)$ by integrating:

$$vy = \int vQ(x)dx.$$

   not forgetting the constant of integration.
3. Divide by v to get the solution:

$$y = \frac{\int vQ(x)dx}{v}.$$

Example

Solve the equation

$$x\frac{dy}{dx} + y = \sin(x)$$

subject to the initial condition $y(\pi/2) = 1$.

Example

Solve the equation

$$x\frac{dy}{dx} - y = x^3, \quad y(1) = 1.$$

Example (Q3(c), Semester 1, '06/'07)

$$e^x \frac{dy}{dx} + 2e^x y = 1.$$

Example

Solve the following differential equation:

$$\frac{dy}{dx} + \cos(x)y = 2xe^{-\sin(x)}.$$

Exercise

Solve the following differential equations:

(i) $y' + \frac{y}{x} = x^2 - \frac{1}{x}, \qquad y(1) = 1/4.$

(ii) $y' + 2y = e^{-x}.$

(iii) $y' = x^2 + x^2 y$

(iv) $y' + 3xy = x$

(v) $y' = \sin(x)y = 3\sin(2x)$

(vi) $xy' + y = 2x\sin(x)$

(vii) $2xyy' = x^2 + 3y^2$

(viii) $y' + \frac{y}{\tan(x)} = 3x + 1$

See also: Problem Set 5.

Over the past 12 weeks or so, we have covered the following topics

Part 1: Functions, derivatives and integrals

- Functions, including the ideas domain, codomain and range, one-to-one and onto, and the inverse of a function. Even and off functions.
- Limits, e.g., squeeze theorem, and l'Hopital's rule,
- Derivatives, including differentiating the product and ratio of twp functions. The chain rule. Derivatives of trigonometric and inverse trig functions,
- Antiderivatives and integrals; logs and exponentials.
- Euler's formula, and Hyperbolic functions.

Part 2: 2nd order differential equations with constant coefficients

- Solving problems of the form $ay'' + by' + cy = 0$, where

$$D := b^2 - 4ac \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases}$$

- Initial and boundary value problems,
- Problems where the right-hand side is a *polynomial*, *exponential* or *trig function*, or the sum or product of these.
- **Power series solutions**.

Part 3: Integrals

- Evaluating indefinite and definite integrals.
- Fundamental theorem of calculus
- Techniques of integration: Substitutions, *Integration by parts*, Reduction formulae; partial fractions.
- Improper integrals: type 1 and 2, including proving that
  - $\int_1^\infty \frac{1}{x^p}dx \begin{cases} \text{diverges} & \text{for } p \leq 1 \\ \text{converges} & \text{for } p > 1 \end{cases}$
  - $\int_0^1 \frac{1}{x^p}dx \begin{cases} \text{converges} & \text{for } p < 1 \\ \text{diverges} & \text{for } p \geq 1 \end{cases}$
  - The comparison test.

Part 4: First Order Differential Equations

- Separable,
- Homogeneous,
- Linear ($\rightarrow$ Integrating factors)