MA211 – **Problem Set 2**

You don't have to hand up any homework from this Problem Set. However, the in-class test on Wednesday 15/10/08 will include at least one part of Questions 9.1, and will feature a question similar to the exercises from Lecture 10.¡

Q6.1 Evaluate the following integrals. Where appropriate, use the antiderivatives given on p41 of Department of Education's Mathematics Tables.

(i) $6t^2 - 1$, (ii) $\frac{x+3}{x^{3/2}}$

(iii) $\int 6dx$ (iv) $\int x^{-2}dx$

(v) $\int \big(x^2 + \cos(x)\big)dx$ (vi) $\int \cos(t)\tan(t)dt$

(vii) $\int (A + Bx + Cx^2)dx$ (viii) $\int \cos(3x)dx$

Q6.2 (i) Show that, for any constants C_1 and C_2,

$$y(x) = C_1x^2 + C_2x^{-2}$$

is a solution to the differential equation

$$x^3y'''(x) + 6x^2y''(x) = 12y(x).$$

(ii) Write down a 2nd order differential equation that has $f(x) = x^2 - x$ as a solution.

Q6.3 Find solutions to the following differential equations. If possible, gave a particular solution, otherwise, give the general solution.

(i) $y'(t) = x - 2$

(ii) $f'(x) = x^{-2} - x^{-3}$, subject to $f(-1) = 0$.

(iii) $y''(x) = x^3 - 1$, with $y'(0) = 0, y(0) = 8$.

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Q7.1 Find solutions to the following DEs

(i) $y'(t) = x - 2$

(ii) $f'(x) = x^{-2} - x^{-3}$, subject to $f(-1) = 0$.

(iii) $y''(x) = x^3 - 1$, with $y'(0) = 0, y(0) = 8$.

(iv) $f''(t) + f(t) = 0$. (Hint: Trig function)

(v) $f''(t) = 9f(t)$ (Hint: Trig function)

Q7.2 For *each* of the following functions, identify the largest possible domain and corresponding range. Is the function one-to-one, onto, or both? Does the function have an inverse? If so, what is it?

(i) $f(x) = 1/(1 - x)^3$ (ii) $f(x) = 1/(x + 1)^2$.

(iii) $f(x) = \sin^{-1}(x)$ (iv) $f(t) = \log_2(x)$.

(v) $f(x) = a^x$ for $a \in (0, 1)$

(vi) $f(x) = \ln(x)$ (vii) $f(t) = \tan^{-1}(x)$.

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Q8.1 (i) Show that $\cos(\tan^{-1}(x)) = \frac{1}{\sqrt{1+x^2}}$.

(ii) Simplify the expression $\sin(\tan^{-1}(x))$

(iii) Simplify the expression $\cos(2\tan^{-1}(x))$

Q8.2 Show that

(i) $\frac{d}{dx}\Big(\sin^{-1}(x)\Big) = \frac{1}{\sqrt{1-x^2}}$.

(ii) $\frac{d}{dx}\Big(\cos^{-1}(\frac{x}{a})\Big) = \frac{-1}{\sqrt{a^2-x^2}}$.

(iii) $\frac{d}{dx}\Big(\tan^{-1}(\frac{x}{a})\Big) = \frac{1}{\sqrt{a^2+x^2}}$.

Hint: Use that

- $\cos^2(x) + \sin^2(x) = 1$,
- $\sec(x) = 1/\cos(x)$,
- $\sec^2(x) = 1 + \tan^2(x)$.

Q8.3 Use the Euler formula to show the following:

(i) $\sin(x) = \frac{-i}{2}(e^{ix} - e^{-ix})$,

(ii) $\frac{d}{dx}\sin(x) = \cos(x)$

(iii) $\int \sin(x) = -\cos(x) + C$

(iv) $\sin^2(x) + \cos^2(x) = 1$

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Q9.1 Define the hyperbolic functions $\cosh(x)$ and $\sinh(x)$ in terms of e^x and e^{-x}.

(i) Show that $\cosh^2 x - \sinh^2 x = 1$.

(ii) What are the largest possible domain for the functions $f(x) = \sinh(x)$ and $f(x) = \sinh^{-1}(x)$? Sketch their graphs.

(iii) Show that $\sinh(2x) = 2\cosh(x)\sinh(x)$

(iv) Prove that

$$\frac{d}{dx}\left(\sinh^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{a^2+x^2}}.$$

(v) Show that

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y.$$

Q9.2 Show that

(i) $\tanh(x) = \frac{e^{2x}-1}{e^{2x}+1}$

(ii) $\cosh^{-1}(x) = \ln(x + \sqrt{x^2+1})$

(iii) $\frac{d}{dx}\tanh(x) = 1 - \tanh^2(x)$

(iv) $\frac{d}{dx}\tanh^{-1}\left(\frac{x}{a}\right) = \frac{1}{a^2 - x^2}$

(v) $\cosh(2x) = \cosh^2(x) + \sinh^2(x)$

(vi) $\cosh(x) + \sinh(x) = e^x$

(vii) $\cosh(x) - \sinh(x) = e^{-x}$

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Q10.1 Find general solutions to the following differential equations:

(i) $y'' + y' - 6y = 0$.

(ii) $3y'' + y' - y = 0$.

(iii) $y'' + 4y' + 2y = 0$

(iv) $y'' + 2y' = 0$

Q10.2 Find general solutions to the following differential equations:

(i) $\frac{3}{4}y'' + 3y' + 3y = 0$.

(ii) $y'' - 8y' + 16y = 0$.

Q10.3 Solve the following differential equations:

(i) $y'' = -2y$.

(ii) $y'' + 4y' + 13y = 0$.

(iii) $y'' + 2y' + 5y = 0$.

(iv) $8y'' + 12y' + 5y = 0$.

Q10.4 Suppose we wish to find the general solution to the DE

$$ay'' + by' + cy = 0 \text{ with } b^2 < 4ac.$$

The roots of the auxiliary equation are $R = k \pm i\omega$ where $k = -b/(2a)$ and $\omega = \sqrt{4ac - b^2}/(2a)$.

Show that if $y(x) = e^{kx}u(x)$ then $u(x)$ satisfies

$$u''(x) + \omega^2 u(x) = 0.$$

Solve this equation to give an expression for $y(x)$. (Note: This is an alternative approach to using Euler's Formula, as was done in class).