Mathematical and **Logic**al Aspects of Computing (CS304/CS310)

http://www.maths.nuigalway.ie/CS304

Lecture 7: Semantic Tableau

Tuesday, 25^{th} Sep 2012

Recall...

Over the past two weeks we introduced some terminology:

- A proposition A is *satisfiable* is there is some interpretation v such that $v(A) = T$. We call v a *model* for A.
- If a proposition is *not* satisfiable, it is *a contradiction*.
- A proposition A is *valid* (or is a tautology) if, for all interpretations v, $v(A) = T$. We write this as: $\models A$.
- If there is some v such that $v(A) = F$, then we say that A is *not valid* (or "falsifiable"), and we write $\not\models A$
- A set of formulas $\{A_1, A_2, \ldots, A_n\}$ is *(simultaneously) satisfiable*, or "consistent as a collection" if there is some (particular) assignment v such that $v(A_1) = v(A_2) = \cdots = v(A_n) = T$. We say v is a model for the set.
- Let $\mathcal{V} = \{A_1, A_2, \ldots, A_n\}$ be a set of propositions, and C be a proposition. If C is true for every model of $\mathcal{V}$, then *C is a logical consequence of* $\{A_1, \ldots A_n\}$. We write $\mathcal{V} \models C$.
- $\mathcal{V} \models C$ if and only if $\{A_1, A_2, \ldots, A_N, \neg C\}$ is inconsistent.

A logical argument?

Suppose the following:

A_1. Ann or Bob have a lecture at 9am on Monday.

A_2. If Bob has a lecture at 9, so too does Cathal.

A_3. Cathal does *not* have a lecture at 9.

Can we conclude that Ann has a lecture at 9?

Semantic Tableau

In general, writing out a full truth table for all formulae in a set is too time-consuming. "*Semantic tableau*" is a faster method to check for consistency.
First recall what a *literal* is:

A **literal** is an atom (i.e., a Boolean variable) or its negation. We call these *positive literal* and *negative literal* respectively.
For a literal p, the pair $\{p, \neg p\}$ is a *complimentary pair of literals*.
For the proposition/formula A, the pair $\{A, \neg A\}$ is a *complimentary pair of formulas*.

Next we use that every 2-place Boolean operator is either

- A **conjunction**, meaning it is true only if *both* atoms have a particular value. There are sometimes called α-formulas.
- A **disjunction**, meaning it is true if *either* atoms have a particular value. There are sometimes called β-formulas.

Examples:

Semantic Tableau

This also applies 2-place operators to (compound) formulas:

- The **conjunction** of two formulas is satisfiable only if **both** are satisfiable.
- A **disjunction** of two formulas is satisfiable if *either* are satisfiable.

Examples:

Building the tableau

We can now construct a tableau for testing if a set of formulas is consistent as a collection:

- List each of the formulas as the root of a tree;
- Take each in turn,
  - if it is an disjunction (of literals or formulas) expand as 2 branches, one for for each of its arguments.
  - if it is an conjunction (of literals or formulas) expand as a single branch for each of its arguments.

Sometimes called *"derived" rules*:

Building the tableau

- If at any stage, a branch contains a complementary pair, we "close" that branch.
- Continue until all branches are closed or all formulas are expanded to literals.
- If any branch remains open, then the original set is consistent as a collection, and the tableau provides a model.

Example: For each of the following propositions, write out the tableau and either find a model, or show that it is not valid.

1. $\neg p \wedge p$
2. $\neg q \vee q$
3. $p \wedge (\neg q \vee \neg p)$. ($\longleftarrow$ Source: Ben-Ari, Section 2.6)
4. $a \wedge \neg\Big((b \wedge c) \to a\Big)$ ($\longleftarrow$ Source: Burris, Exer 2.2.2)
5. $\Big(a \vee (b \to (c \wedge d))\Big) \vee (a \to b)$. ($\longleftarrow$ Source: Burris, Exer 2.2.2)

Tableau for collections of propositions

Suppose that we wish to check if the collection $\{A_1, A_2\}$ is consistent. That is equivalent to showing that $A_1 \wedge A_2$ is valid. Therefore, we can apply the tableau method to establishing the consistency of collections.
Example 1: Is the set $\{p \wedge \neg q, p \to q\}$ consistent?

Example 2: Is the set $\{a \oplus b, b \to c, a \wedge \neg c\}$ consistent? If so, give a model for the set.