CS304/CS310
Mathematical and **Logic**al Aspects of Computing

Lecture 5: Functional completeness; Tautologies and Contradictions

Tuesday,
17$^{\text{th}}$ September 2013

Today's topics:

1. Disjunctive Normal Form (again)
2. Functional completeness
3. Tautologies and Contradictions.

In Lecture 4, we saw how to construct a Boolean Expression that corresponded to some given logic table. The express was in **Disjunctive Normal Form** (DNF): the *disjunction of conjunction of literals*. (Here a "literal" means an atomic proposition, or its negation)

Example: Write the following Boolean Expression in DNF:

$$\underbrace{(a \to b)}_{\mathcal{E}_1} \wedge \underbrace{(b \to c)}_{\mathcal{E}_2}.$$

a	b	c	$a \to b$ ($\neg a \vee b$)	$b \to c$	$\mathcal{E}_1 \wedge \mathcal{E}_2$
F	F	F	T	T	T
F	F	T	T	T	T
F	T	F	T	F	F
F	T	T	T	T	T
T	F	F	F	T	F
T	F	T	F	T	F
T	T	F	T	F	F
T	T	T	T	T	T

$(\neg a \wedge \neg b \wedge \neg c)$

$\vee\ (\neg a \wedge \neg b \wedge c)$

$\vee\ (\neg a \wedge b \wedge c)$

$\vee\ (a \wedge b \wedge c)$

$\oplus$ = "exclusive OR)

Example: Write the following Boolean Expression in DNF:

$$(a \oplus b) \wedge \neg c.$$

a	b	c	$a \oplus b$	$(a \oplus b) \wedge \neg c$	$\neg a \wedge b \wedge \neg c$	$a \wedge \neg b \wedge \neg c$
F	F	F	F	F	F	F
F	F	T	F	F	F	F
F	T	F	T	T	T	F
F	T	T	T	F	F	F
T	F	F	T	T	F	T
T	F	T	T	F	F	F
T	T	F	F	F	F	F
T	T	T	F	F	F	F

So the DNF for $(a \oplus b) \wedge \neg c$ is

$$(\neg a \wedge b \wedge \neg c) \vee (a \wedge \neg b \wedge \neg c)$$

Definition

A set of Boolean operators is said to be *functionally complete* if it can be used to express all possible Boolean expressions. (Note: In Section 2.4 of Ben-Ari's "Mathematical Logic for CS" such sets are said to be "adequate".)

Proposition

The set $\{\wedge, \vee, \neg\}$ is functionally complete.

Proof: [take notes...] Every Boolean Expression is represented as a logic table. Every logic table can be expressed in Disjunctive Normal Form

Since DNF is expressed using AND, OR and NOT, the set {AND, OR, NOT} is funcationally complete.

Now that we know that the set $\{\wedge, \vee, \neg\}$ is *functionally complete*, we can show that both the sets

$$\{\wedge, \neg\} \qquad \text{and} \qquad \{\vee, \neg\}$$

are functionally complete.

Example: We know that {AND, OR, NOT} is functionally complete. However, any compound expression

$E_1 \vee E_2$

can be written as $\neg(\neg E_1 \wedge \neg E_2)$. (This comes from de Morgan's Laws: $\neg(E_1 \vee E_2) \equiv \neg E_1 \wedge \neg E_2$.

So {AND, NOT} is functionally complete.

Somewhat remarkably, there are even smaller sets that are functionally complete. One example is the set that consists just the NAND gate: $\{\uparrow\}$.

Recall that $a \uparrow b \equiv \neg(a \wedge b)$. We know that

a	b	$a \uparrow b$
F	F	T
F	T	T
T	F	T
T	T	F

$\{\neg, \wedge\}$ is F.C.

a	$a \uparrow a$
F	T
T	F

So $\neg a = a \uparrow a$.

That is, we can express NOT using NAND. Also

$a \uparrow b = \neg(a \wedge b)$

So $(a \uparrow b) \uparrow (a \uparrow b) = \neg\neg(a \wedge b) = a \wedge b$. So we can write AND using NAND

One can also show that the set $\{\downarrow\}$ is functionally complete. And (again, remarkably) these are the only two operators with this property.

The following concept is one of the most central to this course:

Definition (Satisfiable)

A proposition P is *satisfiable* if and only if there is an interpretation v such that $v(P) = T$. We say that v is a *model* for P.
If P is *not satisfiable*, then we say

- P is contradictory
- P unsatisfiable
- P is a contradiction.

Examples: $a \wedge b$ is satisfiable: take $a = T$, $b = T$.

$\neg(a \to b)$ is satisfiable: $a = T$, $b = F$

$a \wedge \neg a$ is not satisfiable.

Definition (Valid (= Tautology))

A proposition P is *valid* if and only if $v(P) = T$ for all interpretations v. We often express this as "P is a tautology". We can write this as:

$$\models A.$$

If P is *not valid*, i.e., there is some v such that $v(A) = F$, then we say P is *falsifiable*, and we write $\not\models A$

Examples:

a OR not A is VALID.

$a \wedge \neg b$ is falsifiable (eg $a = F$)

$a \oplus b$ is falsifiable eg $a = T$, $b = F$

$(a \to b) \vee (b \to a)$ is <u>Valid</u>

The relationship between these concepts is:

- P is valid if and only if $\neg P$ is unsatisfiable.
- P is satisfiable if and only if $\neg P$ is falsifiable.

$\Rightarrow$ Suppose P is valid. So $v(P) = T$ for all v. So $v(\neg P) = F$ for all v. So $\neg P$ is unsatisfiable.

$\Leftarrow$ If $\neg P$ is unsatisfiable, there is no v so that $v(\neg P) = T$ so $v(\neg P) = F$ for all v. So $v(P) = T$

stopped here.

More examples (CS304, Summer 11/12, Q2(c)) For each of the following propositions, determine if it is satisfiable or not satisfiable, and if it is valid or not valid.

(iii) $(a \wedge \neg b) \rightarrow (a \vee \neg b)$

(iv) $(a \uparrow b) \rightarrow (a \downarrow b)$.

1. Express the following in DNF:
   (i) $(a \rightarrow b) \vee (b \rightarrow \neg a)$ (ii) $(a \vee b \vee c) \wedge (\neg b \vee \neg c)$
2. Show that the following sets are functionally complete.
   (i) $\{\neg, \rightarrow\}$ (ii) $\{\downarrow\}$
3. For each of the following sets, determine if it is functionally complete or not. Explain your answer.
   (i) $\{\neg, \leftrightarrow\}$
   (ii) $\{\wedge, \vee\}$
   (iii) $\{\rightarrow, T, F\}$ (Note: here we are treating T and F as "nullary operators": operators that take no arguments.
4. Show that
   (i) $((p \oplus q) \oplus q) \equiv p$
   (ii) $((p \leftrightarrow q) \leftrightarrow q) \equiv p$
5. Show that the negation operator cannot be constructed using $\{\wedge, \vee\}$.
6. For each of the following, classify it as *satisfiable*, *not satisfiable* (=contradiction), *valid* (=tautology), *not valid* (note: more than one of these could apply).
   (i) $a \wedge \neg a$ (ii) $\neg a \vee a$ (iii) $\neg\neg a$
   (iv) $a \rightarrow \neg a$ (v) $a \rightarrow a$ (vi) $(\neg a \wedge (a \rightarrow b)) \rightarrow \neg b$