CS304 – CS310

Mathematical and *Logic*al Aspects of Computing

# Lecture 7: Logical consequences (again)

Tuesday, $24^{\text{th}}$ September 2013

**Today's topics**

1. Recall... Logical consequences
2. A valid argument?
3. Semantic Tableau
4. Tableau for collections of propositions
5. Exercises

## Recall... Logical consequences 

In Lecture 6 we first met the following concept:

**Logical Consequence**

Let $\mathcal{V} = \{A_1, A_2, \ldots, A_n\}$ be a set of propositions, and $C$ be a proposition. *$C$ is a logical consequence of $\{A_1, \ldots A_n\}$* iff every model of $\mathcal{V}$ is a model of $C$.
We write $\mathcal{V} \models C$.

$A_1$ $A_2$ $C$

*Example:* Use a logic table to show that $\{a \vee b, \neg b\} \models a$.

| $a$ | $b$ | $A_1$ | $A_2$ | $C$ |
|---|---|---|---|---|
| F | F | F | T | F |
| F | T | T | F | F |
| T | F | T | T | T |
| T | T | T | F | T |

It is possible for $a \vee b$, $b$ true but $a$ false.

*Example:* Use a logic table to show that $\{a \vee b, b\} \not\models a$.

| $a$ | $b$ | $a \vee b$ | $b$ | $a$ |
|---|---|---|---|---|
| F | F | F | F | F |
| F | T | T | T | F |
| T | F | T | F | T |
| T | T | T | T | T |

This shows that $\{a \vee b, b\} \not\models a$.

*Example:* Show that $\{a \vee b, a \to c, b \to c\} \models c$.

| | $a$ | $b$ | $c$ | $a \vee b$ | $a \to c$ | $b \to c$ | $c$ |
|---|---|---|---|---|---|---|---|
| | F | F | F | F | T | T | F |
| | F | F | T | F | T | T | T |
| | F | T | F | T | T | F | F |
| * | F | T | T | T | T | T | T ✓ |
| | T | F | F | T | F | T | F |
| * | T | F | T | T | T | T | T ✓ |
| | T | T | F | T | F | F | F |
| * | T | T | T | T | T | T | T ✓ |

**More examples:** What do the expressions $R \models F$ and $\top \models S$ mean? (Here, as usual, $\top$ is "true" and $F$ is "false").

$R \models F$. So, whenever $R$ is true, so too is $F$

Since $F$ (false) is never true, $R$ is alway false. $R$ is <u>not</u> satisfiable.

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$T \models S$ means, whenever $T$ is true, so too is $S$. So $S$ is always true (ie, $S$ is valid).

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Short-hand: write $\models S$ for $T \models S$

$R \models$ for $R \models F$

## Observation 1.

It is important to note that, if $\mathcal{V} = \{A_1, A_2, \ldots, A_N\}$

$$\mathcal{V} \models C \text{ if and only if } T \models (A_1 \wedge A_2 \wedge \cdots \wedge A_N) \to C.$$

If $V \models C$, then, whenever $A_1, A_2, \ldots, A_N$ are all true so too is $C$.

That is, if $A_1 \wedge A_2 \wedge \cdots \wedge A_N = T$ so too is $C$

So $A_1 \wedge A_2 \wedge \cdots \wedge A_N \to C$.

**Observation 2.**

$\mathcal{V} \models C$ if and only if $\{\underbrace{A_1, A_2, \ldots, A_N}_{\mathcal{V}}, \neg C\}$ is inconsistent.

$\mathcal{V} \models C$ means $\{A_1, A_2, \ldots, A_N\} \models C$

So, whenever $A_1, A_2, \cdots, A_n$ are all true,

So too is $C$.

So if $A_1, A_2, \ldots, A_N$ all true, $\neg C$ is

false.

So $\{A_1, A_2, \ldots, A_N, \neg C\}$ cannot all be true at the same time.

## A valid argument? 

Suppose the following:

$A_1$. Ann or Bob have a lecture at 9am on Monday.

$A_2$. If Bob has a lecture at 9, so too does Cathal.

$A_3$. Cathal does *not* have a lecture at 9.

Can we conclude that Ann has a lecture at 9?

a = "Ann has a 9am lecture"

b = "Bob has a 9am"

c = "Cathal ... "

$A_1$ : $a \vee b$ $\quad$ $A_2$ $b \to c$ $\quad$ $A_3$ : $\neg c$

Argument is $\{a \vee b, b \to c, \neg c\} \models a$.

In general, writing out a full truth table for all formulae in a set is too time-consuming. "*Semantic tableau*" is a faster method to check for consistency.

First recall what a *literal* is:

A **literal** is an atom (i.e., a Boolean variable) or its negation. We call these *positive literal* and *negative literal* respectively.
For a literal $p$, the pair $\{p, \neg p\}$ is a *complimentary pair of literals*.
For the proposition/formula $A$, the pair $\{A, \neg A\}$ is a *complimentary pair of formulas*.

Next we use that every 2-place Boolean operator is either

- A **conjunction**, meaning it is true only if *both* atoms have a particular value. There are sometimes called $\alpha$-formulas.
- A **disjunction**, meaning it is true if *either* atoms have a particular value. There are sometimes called $\beta$-formulas.

**Examples:**

$a \to b \equiv \neg a \vee b$ a disjunction.

$a \downarrow b = \neg(a \vee b)$

$= \neg a \wedge \neg b.$ a conjunction

## Semantic Tableau 

This also applies 2-place operators to (compound) formulas:

- The **conjunction** of two formulas is satisfiable only if **both** are satisfiable.
- A **disjunction** of two formulas is satisfiable if *either* are satisfiable.

**Examples:**

Stopped here, but don't forget the exercises on the last slide!

## Exercises 

Q1. For each of the following, use a truth table to establish if it a is correct logical consequence.

(i) $\{a, c\} \models (a \vee b) \vee (b \vee c)$
(ii) $\{q, \quad \neg p \rightarrow \neg q\} \models p$
(iii) $\{a \rightarrow b, \quad b \rightarrow c\} \models a \rightarrow c$
(iv) $\{a \rightarrow \neg b, \quad b \rightarrow \neg c\} \models \neg a \rightarrow \neg c$

Q2. Use a tableau to check for consistency of the following sets of propositions.

(i) $\{a \vee b, \neg b, \neg a\}$.
(ii) $\{a \vee b, b, \neg a\}$.
(iii) $\{a \vee b, a \rightarrow c, b \rightarrow c, \neg c\}$.

*These three questions are related to examples from earlier in the lecture.*