CS304 – CS310

Mathematical and *Logic*al Aspects of Computing

Lecture 9: Using the tableau method

Tuesday, 1^{st} October 2013

Today's topics

1. Recall: Semantic Tableau
2. (Non)uniqueness
3. Soundness and completeness
4. Proof systems
5. Exercises

Last week we introduced the idea of the "*Semantic tableau*". It is an algorithm that an be used in one of two ways:

1. Given a compound proposition, find a model or show that there is no model (in this latter case we say it is *not valid* or *a contradiction*).
2. Given a collection of propositions, finds a model that satisfies all the props, or asserts that there is no model (in this latter case we say the collection is *inconsistent*).

Often, we are not so interested in what that model might be, but rather if it exists of not.
That is, the Tableau answers the question: *"Is the collection consistent, or not?"*.
We could do this by writing out a truth table for the collection, but the tableau is much faster. It works by searching through the collection (or proposition) for *complimentary pairs of literals*.

To construct a tableau for testing if a set of formulas is consistent as a collection:

- List each of the formulas as the root of a tree;
- Take each in turn,
  - if it is an disjunction (of literals or formulas) expand as 2 branches, one for for each of its arguments.
  - if it is an conjunction (of literals or formulas) expand as a single branch for both arguments.

Sometimes called *"derived" rules*:

$a \vee b$: a | b

$a \wedge b$: a, b

$a \to b$: $\neg a$ | b

$a \uparrow b = \neg(a \wedge b) = \neg a \vee \neg b$: $\neg a$ | $\neg b$.

- If a branch contains a complementary pair, we "close" it.
- Continue until all branches are closed or all formulas are expanded as literals.
- If any branch remains open, then the original set is consistent as a collection, and the literals on that branch give a model.

Example (CS304/CS310 Exam 2012/13) *Use the tableau method to establish if the set of propositions*

$F \vee T = T$ $\quad \neg(T \to F) = T$ $\quad F \to F = T$ $\quad \neg(\neg T \vee F) = \neg F = T$.

$$\{\neg a \vee b,\ \neg(b \to c),\ c \to d,\ \neg(\neg a \vee d)\}$$

is satisfiable as a collection. If it is, then give a model.

$\{\neg a \vee b,\ \neg(\neg b \vee c),\ \neg c \vee d,\ \neg(\neg a \vee d)\}$ $\quad x \to y \equiv \neg x \vee y$.

$\{\underset{\Sigma_1}{\neg a \vee b},\ \underset{\Sigma_2}{b \wedge \neg c},\ \underset{\Sigma_3}{\neg c \vee d},\ \underset{\Sigma_4}{a \wedge \neg d}\}$ by de Morgan

$b, \neg c$ $\quad E_2$

$a, \neg d$ $\quad E_4$.

$\neg c \quad | \quad \underline{d}$ × $\quad E_3$

$\underline{\neg a}$ × $\quad | \quad b$ $\quad E_1$

There is an open branch. So the set is consistent.

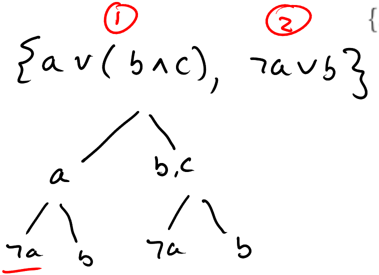
Model: $\{a, b, \neg c, \neg d\}$

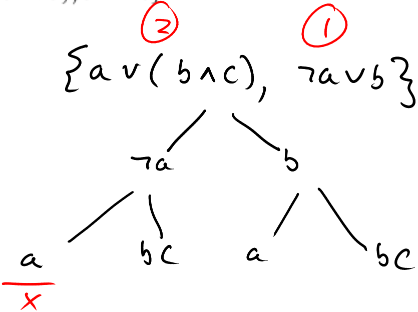
ie $a = T$, $b = T$,
$c = F$ $\quad d = F$

There is no unique tableau for a set of propositions. One may expand the propositions in the set in any order, which can give different tableaux.

Example: Give two different tableaux for the set

$$\{a \vee (b \wedge c), a \rightarrow b\}$$





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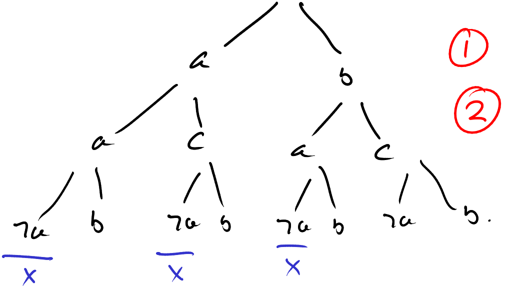
Example: Give two different tableaux for the set

$$\{a \vee (b \wedge c), a \rightarrow b\}$$

$$\{(a \vee b) \wedge (a \vee c), \neg a \vee b\}$$

$$= \{\overset{1}{a \vee b}, \overset{2}{a \vee c}, \overset{3}{\neg a \vee b}\}$$



Let us denote a completed tableau for a proposition P as $\mathcal{T}$.
The following facts are easily established:

1. P is satisfiable if and only if $\mathcal{T}$ is open.
2. P is valid if and only if the tableau for $\neg P$ closes.

1 P is satisfiable $\iff$ there is a model for P $\iff$ there is an open branch in T.

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2. P is valid $\iff$ all assignments make P true
$\iff$ all assignments give $\neg P$ false
$\iff$ There is no model for $\neg P$
$\iff$ Tableau for $\neg P$ is closed (ie all branches closed)

The second of the above points can be written as

P **is valid** $\leftrightarrow$ **tableau for** $\neg P$ **closes.**

This is making two statements:

(Completeness) P **is valid** $\rightarrow$ **tableau for** $\neg P$ **closes.**
(Soundness) **tableau for** $\neg P$ **closes** $\rightarrow$ P **is valid**

"Complete" the tableau will always establish if P is valid.

"Soundness" : never wrong.

Recall (from Lecture 7) the following idea:

> *Let $\mathcal{V} = \{A_1, A_2, \ldots, A_n\}$ be a set of propositions, and C be a proposition. If C is true for every model of $\mathcal{V}$, then C is a logical consequence of $\{A_1, \ldots A_n\}$. We write $\mathcal{V} \models C$.*
> *In many books this is also called a derivation, and we it as "C is derived from $\mathcal{V}$".*

One of the primary uses of the tableau method is to check if the argument

$$\{A_1, A_2, \ldots, A_n\} \models C,$$

is valid by checking if

$$\{A_1, A_2, \ldots, A_n, \neg C\}$$

is (in)consistent.

(a) Use the tableau method to check the validity of

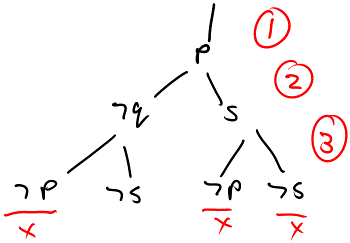
$$\{p, q \to s\} \models p \wedge s.$$

(b) Show that the following is *not* a valid argument:

$$p \to (q \wedge s) \models p \wedge q.$$

Check the consistency of $\{p,\ q \to s,\ \neg(p \wedge s)\}$

$\equiv \{\ p,\ \neg q \vee s,\ \neg p \vee \neg s\ \}$



The Tableau <u>is</u> open. So set is consistent. So argument is <u>not</u> valid

(c) Is the following a valid argument?

- A_1 Arts or Science students take this Logic course;
- A_2 Exactly one of the groups Science and Commerce take logic;
- A_3 If Engineering students take this logic course, then Science students do not;
- Engineering students *do* take this logic course;
- We can conclude that Arts students take logic if and only if Commerce students do too.

Let a be "arts students do Logic"
s be "science " " "
c " " comm " " "
e " eng "

$$\{\underbrace{a \vee s}_{A_1}, \underbrace{s \oplus c}_{A_2}, \underbrace{e \to \neg s}_{A_3}, e\} \models a \leftrightarrow c$$

stopped here

Example: (CS304/310 Exam 2011/12) Determine if the following argument is valid:

> *Wittgenstein visited Mayo or Galway. Wittgenstein did not visit both Mayo and Galway. If Wittgenstein had visited Mayo he would have also visited Galway. Therefore Wittgenstein did not visit Mayo.*

Q1. Use the tableau method to show that the argument $\{p \vee q, p \rightarrow\} \therefore q$ is valid.

Q2. Use the tableau method to check if the following is a valid argument:
If the lecture is given then the students will attend if the lecture is not boring.
If the lecture is given then it will not be boring
Therefore, if the lecture is given, the students will attend.

Q3. Is the following a valid argument?

> *If George Boole was Irish, then he was the first Professor of Mathematics at UCC. George Boole was not Irish. Therefore he was not the first Professor of Mathematics at UCC.*