CS304 – CS310

Mathematical and *Logic*al Aspects of Computing

Lecture 10: Conjunctive Normal Form, ~~and Resolution~~

Friday, 3$^{\mathrm{rd}}$ October 2013

Today's topics

1. An example from Lecture 9...
2. Semantics
3. Towards resolution
4. Conjunctive normal form
5. Clause form
6. ~~Resolution~~
7. Exercises

Here is an example from the end of Lecture 9: **Example: (CS304/310 Exam 2011/12)** (Use the tableau method to) Determine if the following argument is valid:

> 1 *Wittgenstein visited Mayo or Galway.* P_2 *Wittgenstein did not visit both Mayo and Galway.* P_3 *If Wittgenstein had visited Mayo he would have also visited Galway.* C *Therefore Wittgenstein did not visit Mayo.*

m = " W visited mayo". g = " W visited Galway".

$P_1 = m \vee g$ $P_2 = \neg(m \wedge g)$ $P_3 : m \to g$

$C : \neg m$. $\{m \vee g, \neg(m \wedge g), m \to g\} \models \neg m$

Test for consistency of $\{m \vee g, \neg(m \wedge g), m \to g, \neg(\neg m)\}$

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> *Wittgenstein visited Mayo or Galway. Wittgenstein did not visit both Mayo and Galway. If Wittgenstein had visited Mayo he would have also visited Galway. Therefore Wittgenstein did not visit Mayo.*

$\{\underbrace{m \vee g}_{P_1}, \underbrace{\neg(m \wedge g)}_{P_2}, \underbrace{m \to g}_{P_3}, \underbrace{\neg(\neg m)}_{\neg C}\}$

$= \{m \vee g, \neg m \vee \neg g, \neg m \vee g, m\}$.

- m ($\neg C$)
  - $\neg m$ ×
  - g (P_3)
    - $\neg m$ ×
    - $\neg g$ × (P_2)

All closed: So this is a valid argument

An example from Lecture 9... (2/12)

Here is an example from the end of Lecture 9: **Example: (CS304/310 Exam 2011/12)** (Use the tableau method to) Determine if the following argument is valid:

> *Wittgenstein visited Mayo or Galway. Wittgenstein did not visit both Mayo and Galway. If Wittgenstein had visited Mayo he would have also visited Galway. Therefore Wittgenstein did not visit Mayo.*

Here is another way of solving this problem, which relates to our next topic: resolution. [Note: This went horribly wrong when I tried it in class! This should be correct]

We want to verify that every instance where $\{m \vee g, \neg(m \wedge g), m \to g\} = \{P_1, P_2, P_3\}$ are true, so too is $\neg m$

Lets first suppose $m = T$. Then $m \to g$ gives $g = T$.
But then $\neg(m \wedge g) = F$, so there is no need to consider this case further.

Next suppose $m = F$. Then $m \vee g$ gives that $g = T$.
Also $\neg(m \wedge g) = \neg F = T$. So we get all of P_1, P_2 and P_3 true. So too is $\neg m$.

Here is an example from the end of Lecture 9: **Example: (CS304/310 Exam 2011/12)** (Use the tableau method to) Determine if the following argument is valid:

> *Wittgenstein visited Mayo or Galway. Wittgenstein did not visit both Mayo and Galway. If Wittgenstein had visited Mayo he would have also visited Galway. Therefore Wittgenstein did not visit Mayo.*

Next suppose $g = T$. Then, to satisfy $P_2 = \neg(m \wedge g)$ we must have $m = F$, again giving our conclusion.

Finally, if $g = F$, then $P_1 = m \vee g$ gives $m = T$. But then this gives $m \rightarrow g = F$ so we are not concerned with this case.

So we've found that, whenever all of P_1, P_2 and P_3 are true, so too is $\neg m$
as required.

What is “semantic” about the Semantic Tableau?

From the Merrian-Webster dictionary

SEMANTICS:

- *the study of meaning...*
- *the meaning, or an interpretation of the meaning, of a word, sign, sentence, etc.: “Let’s not argue about semantics”.*

From the Oxford Dictionary

Syntax *describes the rules by which words can be combined into sentences, while* **semantics** *describes what they mean.*

Towards resolution (4/12)

The use of logic-table to analyse logical propositions is *semantic*: it deals with (all) possible interpretations of the propositions.
Also, the tableau method is *semantic* since it is an algorithm that finds a particular interpretation that satisfies a proposition (or shows none exists).
Alternatively, we can use a *syntactic* approach.

The classic example is: $\{p \vee q, \neg p \vee q\} \models q.$

Semantic

p	q	$p \vee q$	$\neg p \vee q$	q
F	F	F	T	F
F	T	T	T	T
T	F	T	F	F
T	T	T	T	T

Sytactic.

If $p = T$, and $\neg p \vee q = T$

Then $\neg p \vee q = F \vee q = q$.

So $q = T$.

otherwise, $p = F$. So

$p \vee q = F \vee q = q$. So q = true

Recall: *Disjunctive Normal form*, where we wrote a compound proposition is the **disjunction of conjunctions**.
Now we want to write propositions as the **conjunction of disjunctions**.
This is called "*Conjunctive normal form*" (CNF). When a compound proposition in CNF is expressed as a set of propositions with implied conjunction, this is called "*Clause form*".

Example:

$(p \vee q) \wedge (\neg p \vee q)$ is CNF

$\{p \vee q, \neg p \vee q\}$ is Clause form.

$(a \vee \neg b \vee c) \wedge (\neg a \vee \neg b \vee c)$ is CNF.

But $(a \wedge b) \vee c$ is <u>Not</u> CNF

$\equiv (a \vee c) \wedge (b \vee c)$ <u>is</u> in CNF

neither is $(a \to b) \wedge (b \to c)$.

But $(\neg a \vee b) \wedge (\neg b \vee c)$ is CNF

Conjunctive normal form (6/12)

(This slide is based on Ben-Ari, Chapter 4) When we studied Disjunctive Normal Form (DNF) in Lecture 4, we did so from a Truth Table. We could use that approach with Conjunctive Normal Form (CNF) too. But instead we'll deduce the CNF as follows

1. Rewrite all operators using just AND, OR and NOT.
2. Use de Morgan's laws so that any negation applies to atomic propositions, not compound propositions.
3. Eliminate double negations.
4. Use the distributive laws:

$$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$$

$$(A \wedge B) \vee C \equiv (A \vee C) \wedge (B \vee C)$$

Conjunctive normal form

(a) Write $a \leftrightarrow b$ in both **Disjunctive Normal Form** (DNF) and **Conjunctive Normal Form** (CNF).

(b) Write $a \oplus b$ in both **Disjunctive Normal Form** (DNF) and **Conjunctive Normal Form** (CNF). ($\longleftarrow$ Exer)

(c) Write $(\neg a \to \neg b) \to (a \to c)$ in CNF.

Finished here

Exercises

Q1. Write the following in Conjunctive Normal Dorm, ~~an~~ and then express in *clause form*

1. $a \uparrow b$
2. $\big((a \downarrow b) \rightarrow c\big) \vee (c \rightarrow (a \wedge b)$

Q2. Use resolution to show that the following arguments are valid:

1. $\{\neg a \vee (b \wedge c), b \rightarrow (a \wedge \neg c)\} \therefore \neg a$
2. $\{p \vee q, \quad p \rightarrow r, \quad q \rightarrow r\} \therefore r.$