CS304/CS310 (Logic): Problem Set 1

Assignment: Q1(i), Q4 (iv)+(v); Q6; Q10(i)+(ii); Q12(iii). Deadline: Oct 8th.

1. Recall the "half adder" discussed in Lecture 2.
   (i) $\star$ Show how to construct it using only $\uparrow$.
   (ii) Show how to construct it using only $\downarrow$.

2. Find out what a "full adder" is. Show how to construct it using the connectives $\{\wedge, \vee \neg\}$.

3. Prove the following identities:
   (i) $a \wedge b \equiv b \wedge a$ (ii) $a \vee b \equiv b \vee a$
   (iii) $\neg(a \vee b) \equiv \neg a \wedge \neg b$
   (iv) $(a \vee b) \vee c \equiv a \vee (b \vee c)$
   (v) $(a \wedge b) \vee c \equiv (a \vee c) \wedge (b \vee c)$
   (vi) $(a \vee b) \wedge c \equiv (a \wedge c) \vee (b \wedge c)$

4. Use truth tables to prove or disprove the following identities:
   (i) $\big((a \oplus b) \oplus b\big) \equiv a$
   (ii) $\big((a \leftrightarrow b) \leftrightarrow b\big) \equiv b$
   (iii) $a \wedge (\neg a \vee b) \equiv (a \wedge b)$
   (iv) $\star(a \wedge b) \vee (a \wedge \neg b) \equiv a$
   (v) $\star(a \vee b) \wedge (\neg a \vee c) \equiv (b \vee c)$
   (vi) $a \to b \equiv = \neg a \vee b$. Give an equivalent Boolean express that uses only **NOT** and **AND**.
   (vii) $a \leftrightarrow b \equiv (a \to b) \wedge (b \to a)$.
   (viii) $a \oplus b = (a \vee b) \wedge (\neg a \vee \neg b)$

5. (Based on Ben-Ari, Q2.9) What is meant by the statement: $P \models Q$? Show that
$$(a \wedge b) \to c \models (a \to c) \vee (b \to c).$$
   Given an explanation (in English) for what this means.

6. $\star$ NUI Galway has a rule which states that, in order to successfully complete a first year module, students must score at least 35% for their homework assignments. When questioned, four students answer
   Adam: I scored 65% for my homework.
   Brian: I successfully completed my 1st year Mathematics module.
   Cara: I did not successfully completed my 1st year Mathematics module.
   Dee: I scored 25% for my homework assignments.

   Who should be quizzed further in order to establish if the rule was applied correctly?

7. Of the following binary operators, which commute? (Recall, the 2-place operator $\star$ commutes if $a \star b \equiv b \star a$).
   (i) exclusive-or $\oplus$
   (ii) NAND $\uparrow$
   (iii) material implication $\to$
   (iv) equivalence $\leftrightarrow$

8. Which of the following associative rules are correct?
   (i) $(a \wedge b) \wedge c \equiv a \wedge (b \wedge c)$
   (ii) $(a \vee b) \vee c \equiv = a \vee (b \vee c)$
   (iii) $(a \to b) \to c \equiv a \to (b \to c)$
   (iv) $(a \leftrightarrow b) \leftrightarrow c \equiv a \leftrightarrow (b \leftrightarrow c)$

9. (i) Show that $\{\neg, \to\}$ is functionally complete.
   (ii) Show that the set $\{\to, \top, F\}$ is functionally complete. (That is, we can form any Boolean operator using just the material implication operator and the constants "true" and "false".

10. (Based on Exer 2.20 of Ben-Ari's *Mathematical Logic for Computer Science*, 3rd Ed.) The ternary "*if-then-else*" (written $ifte(a, b, c)$ or $a?b : c$)

a	b	c	$a?b : c$
F	F	F	F
F	F	T	T
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	F
T	T	F	T
T	T	T	T

   (i) $\star$ Sketch a Venn diagram for this operator.
   (ii) $\star$ Show that $a?b : c \equiv (a \to b) \wedge (\neg a \to c)$.
   (iii) Write $a?b : c$ in Disjunctive Normal Form.
   (iv) Show that the $ifte$ operator on its own is functionally completed (called "an adequate set" by Ben-Ari) if the use of the constants T and F are allowed.

11. Classify each of the following compound propositions as *satisfiable*, *not satisfiable* (=contradiction), *valid* (=tautology), *not valid* (note: could be more than one of these).
   (i) $a \wedge \neg a$ (ii) $\neg a \vee a$
   (iii) $a \to \neg a$ (iv) $(\neg a \wedge (a \to b)) \to \neg b$
   (v) $\big(a \wedge b\big) \to \big(c \vee (\neg b \to \neg c)\big)$

12. For each of the following, use a truth table to establish if it a is correct logical consequence.
   (i) $\{p \to q, \neg q\} \models \neg p$
   (ii) $\{q, \neg p \to \neg q\} \models p$
   (iii) $\star$ $\{a \to b, b \to c\} \models a \to c$
   (iv) $\{a \to \neg b, b \to \neg c\} \models \neg a \to \neg c$