CS304/CS310 – Mathematical and **Logic**al Aspects of Computing

Problem Set 2

Corresponds to Lectures 6–12 (Dr Niall Madden). For further exercises, see the last slide from lectures.

1. For each of the following, determine if it is a correct logical consequence:

   (i) $\{a\} \models a \vee b$

   (ii) $\{a\} \models a \wedge b$.

   (iii) $\{a \vee \neg a\} \models a$.

   (iv) $\{a \wedge b,\ a \vee c\} \models b \vee c$

   (v) $\{a \leftrightarrow b,\ b \wedge c\} \models a \wedge b$

   (vi) $\{\neg a \vee \neg b,\ a,\ b \vee c\} \models c$

   (vii) $\{a \rightarrow (b \vee c),\ \neg a,\ b \rightarrow c\} \models b \wedge c$

2. For each of the following collection of propositions, use the *tableau method* to show that it is consistent, and find an interpretation that satisfies all the propositions in the collection.

   (i) $\{\neg(a \rightarrow b), \quad b \vee c, \quad a \rightarrow c\}$,

   (ii) $\{p \rightarrow (q \vee r), \quad q \rightarrow p, \quad \neg r\}$.

3. Use the tableau method to show that the argument $\{p \vee q, p \rightarrow q\} \models q$ is valid.

4. Use the tableau method to check if the following is a valid argument:
   If the lecture is given then the students will attend if the lecture is not boring.
   If the lecture is given then it will not be boring
   Therefore, if the lecture is given, the students will attend.

5. Use resolution to show that the following arguments are valid:

   (i) $\{\neg a \vee (b \wedge c),\ b \rightarrow (a \wedge \neg c)\} \models \neg a$

   (ii) $\{p \vee q, \quad p \rightarrow r, \quad q \rightarrow r\} \models r$.

6. Use resolution to check if the following argument is valid:

   You will not pass this module if you do not pass the exam.
   You will only pass the module if you have attended enough lectures.
   You have attended enough lectures and passed the exam.
   Therefore you will pass the module.

7. The class rep argues that:
   The 2nd year students or the 3rd year students will go to the party.
   If the 3rd year students go then the 4th year students will go unless the 1st year students go.
   The 1st year students will go if the 2nd year students do not go.
   Therefore the 3rd year students will go. Letting y_1 mean "the 1st year students will go to the party", y_2 mean "the second year students will go", etc., show that this argument can be represented by

   $$\{y_2 \vee y_3,\ y_3 \rightarrow (\neg y_1 \rightarrow y_4),\ \neg y_2 \rightarrow y_1\} \models y_4.$$

   Is this a valid argument?

8. A logician reasons as follows:

   "If the argument is valid and the premisses are true then the conclusion is true.
   The premisses are false but the conclusion is true.
   So the argument is valid."

   Using c, p and v for "the conclusion is true", "the premisses are true" and "the argument is valid", construct the argument which underlies the above reasoning. Test for validity the argument you have constructed.

9. Give counter-examples to each of the following (i.e., find a model for the premises that is not a model for the conclusion):

   (i) $\{a \rightarrow b, a \rightarrow c\} \models b \rightarrow c$.

   (ii) $\{a \rightarrow b, q\} \models p$.

   (iii) $\{p \vee \neg q, r \vee q\} \models \neg p \vee r$.

10. Write the following in both *Conjunctive Normal Form* and *Clause Form.*

    (i) $a \uparrow b$

    (ii) $\big((a \downarrow b) \rightarrow c\big) \vee (c \rightarrow (a \wedge b)$

11. Review your notes for Lecture 11. We gave as Property 5 of Clause forms, than a set containing an empty clause is *not* satisfiable. Show (the somewhat surprising fact) that the empty set of clauses is valid.

    Hints:

    - An example of a set containing an empty clause is $U = \{\{a\}, \{\Box\}\}$.
    - The set $U = \{\{\Box\}\}$ that contains only the empty clause, is *not* the same as the set $V = \{\}$ that contains no clause.

- Suppose $U = \{\{a\}\}$. Apply Property 2.

12. Let U and V be propositions expressed in clause form. Recall that $U \approx V$ means that the U is satisfiable if and only if V is satisfiable.

    For each of the following, determine if $U \approx V$

    (i) $U = \{\{a\}, \{b\}\}$, $V = \{\{ab\}\}$.

    (ii) $U = \{\{a, b\}, \{a, \neg c\}, \{b, \neg c\}\}$, $V = \{\{b, \neg c\}\}$.

    (iii) $U = \{\{a, \neg a\}\}$, $V = \{\{\square\}\}$.

    (iv) $U = \{\{a, \neg a\}\}$, $V = \{\{\ \}\}$.

13. Use resolution to test following argument for validity:

    $$\{p \to (q \to r), \quad (r \wedge s) \to \neg t\} \models p \to (s \to \neg t).$$