2 Matrix multiplication

This lecture follows very closely Lecture 1 of Trefethen and Bau ([TB97]), *Numerical Linear Algebra.* The key idea is to think of matrix-vector multiplication as constructing a linear combinations of the columns of the matrix.

The first 5 chapters of this book are freely available from the author's homepage:
http://www.comlab.ox.ac.uk/nick.trefethen/text.html

The summary below should be read in the context of those notes. I don't reproduce their text, just repeat the examples done in class.

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2.1 Matrix vector multiplication

If $A \in \mathbb{C}^{m \times n}$ and $x \in C^n$ and b is the m-vector given by $b = Ax$, then

$$b_i = \sum_{j=1}^{N} a_{ij} x_j.$$

Example 2.1.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

This gives

$$\mathbf{b} = \begin{pmatrix} x_1 a + x_2 b \\ x_1 c + x_2 d \end{pmatrix} = x_1 \begin{pmatrix} a \\ c \end{pmatrix} x_2 \begin{pmatrix} b \\ d \end{pmatrix}.$$

So $\mathbf{b}$ is a linear combination of the columns of A with coefficients from x.

2.2 Matrix-matrix multiplication

If $A \in C^{m \times n}$, $C \in \mathbb{C}^{n \times p}$ and $B = AC$, then B is the $m \times p$ matrix given by

$$b_{ij} = \sum_{k=1}^{n} a_{ik} c_{kj}.$$

But in keeping with the ideas above, let us consider the formula for column j of B:

$$\mathbf{b_j} = \sum_{k=1}^{n} c_{kj} \mathbf{a_k}.$$

So column j of B is a linear combination of all the columns of A, with the coefficients taken from column j of C.

Other examples that are worth considering, include computing the inner and outer products of the vectors $(a, b, c)^T$ and $(d, e, f)^T$.

2.3 Involutory, projector, and nilpotent matrices

A particularly important case of computing computing matrix products, is computing matrix powers.

For a square matrix, A, we say that

- A is *involutory*, if $A^2 = I$.
- A is a *projector* (or is *indepotent*), if $A^2 = A$.
- A is *nilpotent*, if $A^k = 0$ for some $k > 0$

Example 2.2. Let

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Write down A^2, A^3 and A^4.

2.4 Triangular matrices

Sometimes the structure of a matrix is preserved under matrix multiplication. An obvious example is that the product of two diagonal matrices is diagonal. A more interesting case is:

Example 2.3. Show that the product of two lower triangular matrices is itself lower triangular.

Note: this last example will crop up again. We will see that the processes of Gaussian Elimination involves repeatedly multiplying a matrix by lower triangular matrices in order to yield an upper triangular matrix. This will lead to the idea of factorising a matrix as the produce of lower and upper triangular matrices.

2.5 Exercises

Do Exercises (i)–(viii) in Section 1.7 of Ipsen's *Numerical Matrix Analysis: Linear Systems and Least Squares.*