3 Matrix inverse

3.1 Range and Rank

The *range* of a matrix is the set of vectors that can be expressed as Ax for some x. We now understand that this means that the range of a matrix is the same as the space spanned by its columns, which we call the *column space*. (The *row space* is the space spanned by the (row) vectors that for its rows).

The nullspace of a matrix is set of vectors $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$. Since we think of matrix-vector multiplication as forming a linear combination of the columns of A, if the columns of A are linearly independent then the null space contains only the zero vector.

The column (row) rank of A is the dimension of the column (row) space. A slightly surprising fact that the row rank and column rank of a matrix are the same. We'll leave the proof until after we know what the SVD is. But in the mean time, we'll note that if the rank of $A \in C^{m\times n}$ is $\min(m, n)$ then it has *full rank*.

This will lead us to looking the idea of invertible matrices.

3.2 Matrix Inverses

One can consider the idea of solving the linear system $Ax = b$ as a *change of basis operation* (See [TB97, Lecture 1]).

We then went on to think about inner products and orthogonality.

. . . .

Recall that the rank of the matrix is the number of linearly independent rows/columns that it contains.

If $A \in \mathbb{R}^{n\times n}$ has full rank, then any vector $\mathbf{b} \in \mathbb{R}^n$ can be expressed as a linear combination of the columns of A. Or, to put it another way, the problem $A\mathbf{x} = \mathbf{b}$ has a solution. Or yet another way: there exists a matrix $A^{-1} \in \mathbb{R}^{n\times n}$ such that $\mathbf{x} = A^{-1}\mathbf{b}$.

Denoting as usual $I = (e_1|e_2|\cdots|e_n)$, given a matrix $A \in \mathbb{R}^{n\times n}$, if there exists $Z \in \mathbb{R}^{n\times n}$ such that $AZ = ZA = I$, then we say that A is *invertible* or *nonsingular*. And we write $Z = A^{-1}$.

There are several other statements that are equivalent to saying that $A \in \mathbb{C}^{m\times m}$ is invertible. These include

- $\text{range}(A) = \mathbb{R}^n$.
- $\text{null}(A) = \{\mathbf{0}\}$.
- The set of eigenvalues of A does not include zero.
- The set of singular values of A does not include zero.
- $\det(A) \neq 0$.
- ...

3.3 Solving linear equations

When trying to solve $A\mathbf{x} = \mathbf{b}$, we never, ever (well, hardly ever) first try to find A^{-1} so that we can compute $\mathbf{x} = A^{-1}\mathbf{b}$.

One of the key approaches is to find matrices L and U which are (respectively) unit lower and upper triangular matrices such that $A = LU$. We'll cover this in lectures soon (though we'll emphasise a special variant for symmetric positive definite matrices: Cholesky Factorisation).

For now we'll think of solving a linear system as a change of basis operation: we are looking for the expansion of b in the column of A.

This is not to say that there aren't many cases where we are interested finding A^{-1}. But usually only in cases where this is easy, for example if A is *unitary*...