5 Norms (Part 2)

5.1 Inner Products and norms

From Cauchy-Schwarz, it follows that

$$-1 \leqslant \frac{x^T y}{\|x\|_2 \|y\|_2} \leqslant 1, \quad \text{for any } x, y \in \mathbb{R}^n.$$

As we all know, this quantity can be thought of as the cosine of the angle between the vectors x and y. So, for example, if x is *orthogonal* to y, then $x^T y = 0$.

5.2 Subordinate Matrix Norms

Definition 5.1. Given any norm $\|\cdot\|$ on $\mathbb{R}^n$, there is a *subordinate matrix norm* on $\mathbb{R}^{m\times n}$ defined by

$$\|A\| = \max_{x\in\mathbb{R}^n_\star} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\|, \tag{5.1}$$

where $A \in \mathbb{R}^{m\times n}$ and $\mathbb{R}^n_\star = \mathbb{R}^n/\{\mathbf{0}\}$.

The motivation for this definition is that we like to think of $A \in \mathbb{R}^{m\times n}$ as an *operator* or linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. So rather than the norm giving us information about the "size" of the entries of a matrix, it tells us how much the matrix can change the size of a vector. However, this definition does not tell us *how* to compute a matrix norm in practice. That is what we'll look at next.

5.3 The 1-norm on $\mathbb{R}^{m\times n}$

Theorem 5.2.

$$\|A\|_1 = \max_{j=1,\dots,n} \sum_{i=1}^n |a_{i,j}|. \tag{5.2}$$

The proof is reasonably easy if we think of matrix-vector multiplication as forming a linear combination of the columns of A.

5.4 The max-norm on $\mathbb{R}^{n\times n}$

Theorem 5.3. *For any $A \in \mathbb{R}^{n\times n}$ the subordinate matrix norm associated with $\|\cdot\|_\infty$ on $\mathbb{R}^n$ can be computed by*

$$\|A\|_\infty = \max_{i=1}^{n} \sum_{j=1}^n |a_{ij}|.$$

That is, the ∞-norm of a matrix is just largest absolute-value row sum of the matrix

5.5 The 2-norm on $\mathbb{R}^{n\times n}$

It turns out that the 2-norm of a matrix is the square root of the largest eigenvalue of $B = A^T A$. For this to make sense, we should note that all the eigenvalues of $A^T A$ are real and be nonnegative. *Why?*

We postpone a detailed discussion until after we've met the *singular value decomposition*.

5.6 Consistency of matrix norms

It should be clear from (5.1) that, if $\|\cdot\|$ is a subordinate matrix norm, then

$$\|Au\| \leqslant \|A\|\|u\|,$$

for *any* $u \in \mathbb{R}^n$. The analogous statement from the product of two matrices is:

Definition 5.4. A matrix norm $\|\cdot\|$ is *consistent* if

$$\|AB\| \leqslant \|A\|\|B\|, \qquad \text{for all } A, B \in \mathbb{R}^n.$$

Theorem 5.5. *Any subordinate matrix norm is consistent.*

Not every matrix norm is norm is Consistent. See Exercise 5.8.

5.7 Some exercises on norms

Exercise 5.1. Given an norm on $\mathbb{R}^n$, denoted $\|\cdot\|$ we can define the *weighted norm* $\|\cdot\|_A$

$$\|x\|_A := \|Ax\|,$$

where A is an invertable matrix. Show that this is indeed a norm on $\mathbb{R}^n$.

If $\|\cdot\|$ is a matrix norm on $\mathbb{R}^{n\times n}$, is it true that

$$\|B\|_A := \|AB\|,$$

is a norm? Explain you answer.

Exercise 5.2. Show that for any of subordinate matrix norm, $\|I\| = 1$. (That is, the norm of the identity matrix is 1).

Exercise 5.3. Show that the eigenvalues of a hermitian matrix are purely real.

Exercise 5.4. In some books also give a definition of a norm that includes that $\|x\| \geqslant 0$ for all x. Should we add this to Definition 4.2, or can it be deduced?

Exercise 5.5. Show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms.

Exercise 5.6. Prove Theorem 5.3.

Exercise 5.7. Prove Theorem 5.5.

Exercise 5.8. Suppose we define a different "max" norm of a matrix as follows:

$$\|A\|_{\widetilde{\infty}} = \max_{i,j} |a_{ij}|.$$

Show that $\|A\|_{\widetilde{\infty}}$ is a norm. Then find an example of a pair of matrices $A, B \in \mathbb{R}^{n\times n}$ such that $\|AB\|_{\widetilde{\infty}} > \|A\|_{\widetilde{\infty}}\|B\|_{\widetilde{\infty}}$.