6 Unitary matrices and matrix norms

6.1 Unitary Matrices

(See [TB97, Lecture 2].)

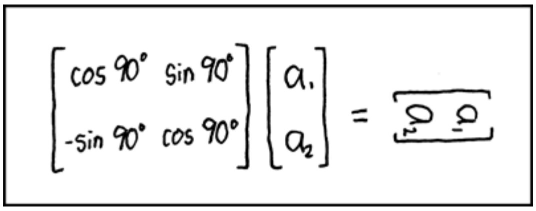
A square matrix $Q \in \mathbb{C}^{m\times m}$ is *unitary* if $Q^{-1} = Q^\star$. The columns of a unitary matrix form an *orthonormal basis* for $\mathbb{C}^m$.

Unitary matrices form an important part of many algorithms in numerical linear algebra, particularly ones for revealing eigenvalues.

They can be thought of as "change of basis operators". Suppose $Qx = b$. So if we express b as a linear combination of the columns of Q, then the coefficients are from x. Conversely, if we express x as a linear combination of the columns of $Q^\star$, then the coefficients are from b.

Other important facts include that transformation by a unitary matrix preserves the angles between vectors. It also preserves length of vectors in the 2-norm.

We'll look at some examples of unitary matrices, and then return move on to singular values.



6.2 General Matrix Norms

(See Trefethen and Bau, Lecture 3).
We'll now return to the idea of a matrix norm. We defined subordinate matrix norms in Definition 5.1. However, since the set of matrices $\mathbb{C}^{m\times n}$ forms a vector space, any function that maps such a matrix to a real number, and satisfies the conditions in Definition 4.1 is a (matrix) norm.

In Exercise 5.8, we saw an example of a silly matrix norm. A much more sensible, and important, example is the *Frobenius* norm:

$$\|A\|_F := \Big(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\Big)^{1/2}. \qquad (6.1)$$

Some facts about $\|\cdot\|_F$.

- The *trace* of $A \in \mathbb{C}^{m\times m}$ is the sum of its diagonal entries:
$$\mathrm{tr}(A) = \sum_{k=1}^{m} a_{kk}.$$
This is used in the definition of the *trace inner product*: $(A, B)_{\mathrm{tr}} = \mathrm{tr}(A^\star B)$. Moreover, $\|A\|_F = \sqrt{(A, A)_{\mathrm{tr}}}$.
- The Frobenius norm is consistent. (See Exercise 6.2).
- If Q is unitary, then $\|QA\|_2 = \|A\|_2$ and $\|QA\|_F = \|A\|_F$. (See also Exercise 6.4).

6.3 Exercises

Exercise 6.1. From the examples of unitary matrices that we looked at in class it seems that, if Q is unitary, then

(i) $\det(Q) = 1$,

(ii) If λ is an eigenvalue of Q, then $|\lambda| = 1$.

Prove these statements, or give counter-examples to them.

Exercise 6.2. Show that the Frobenius norm is consistent. (Hint: using the definition in (6.1), and the Cauchy-Schwarz inequality).

Exercise 6.3. It is not hard to show that, if Q is unitary, then $\|Q\|_2 = 1$: just recall from our earlier discussion that $\|Q\mathbf{x}\|_2 = \|\mathbf{x}\|_2$. What can we say about $\|Q\|_F$?

Exercise 6.4. Suppose that Q is unitary.
Prove that $\|AQ\|_2 = \|A\|_2$.
Prove that $\|AQ\|_F = \|A\|_F$.
Is it true that $\|AQ\|_1 = \|A\|_1$?