7 The singular value decomposition

7.1 Singular Values

The singular values $\{\sigma_1, \sigma_2, \dots, \sigma_m\}$ of a matrix A can be defined in several ways:

- the σ_i^2 are the eigenvalues of $B = A^\star A$,
- σ_i is the length of the semiaxis of the hyperellipse that is the image of the unit circle under the linear transformation defined by A.
- Every matrix A has a factorisation $A = U\Sigma V^\star$ where U and V are unitary matrices, and Σ is a diagonal matrix. Then the enties of Σ are the singular values.

Please read Lecture 4 of Trefethen and Bau. T+B make the important observation in their introduction: *the image of the unit sphere under any* $m \times n$ *matrix is a hyperellipse.*

Suppose that $\{\|u_i\|\}_{i=1}^m$ is a set of orthonormal vectors, such that principal semiaxes of the hyperellipse are $\{\sigma_i u_i\}$.

These scalars $\sigma_1, \sigma_2, \dots, \sigma_m$ are the *singular values of* A, and encapsulate many important properties of A.

7.2 Singular Value Decomposition

This exposition follows Lecture 4 of Trefethen and Bau pretty closely. We first see the SVD of $A \in \mathbb{C}^{m\times n}$ as

$$AV = \Sigma U,$$

where

- $U = (u_1|u_2|u_3|\dots|u_m)$, the matrix that has as its columns the orthoronormal vectors in the direction of the principal semiaxes of the image of the unit sphere $S \in \mathbb{R}^n$.
- $\Sigma = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$ is the diagonal matrix containing the lengths of the principal semiaxes. We order them as
$$\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n \geqslant 0.$$
- Then $V = (v_1|v_2|v_3|\dots|v_n)$ is the matrix whose columns are the preimages of the principal semiaxes, i.e., $Av_i = \sigma_i u_i$. It should be clear that the $\{v_i\}$ form an orthonormal set, so $V^{-1} = V^\star$.

7.3 Examples

We'll look at some examples in class, using a Matlab script. One of these will be based on the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

In Figure 7.1 we show S, the unit circle in $\mathbb{R}^2$. In Figure 7.3 is its image under A. In between are shown VS (left in Figure 7.2), which as you can see causes a rotation of S, but no stretching (because V is unitary, $\|x\|_2 = \|Vx\|_2$; and ΣVS (bottom), which does the stretching. Multiplication by U again rotates giving the final image.

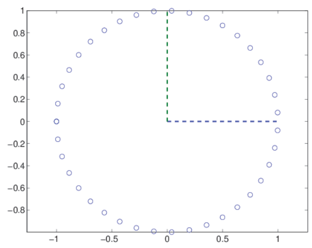

Figure 7.1: The unit circle S in $\mathbb{R}^2$

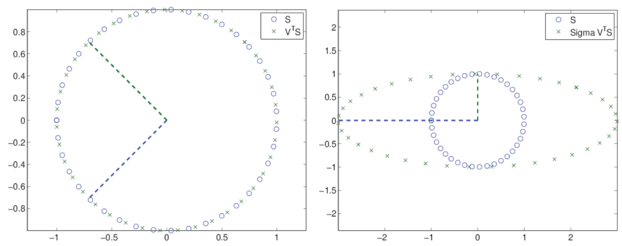

Figure 7.2: VS (left) and ΣVS

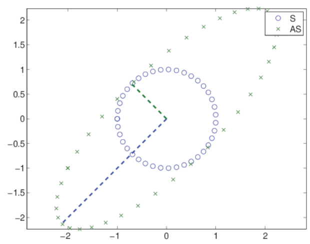

Figure 7.3: AS, the image of the unit circle

The SVD for this is

$$A = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$