8 One theorem about the SVD

This class started very slowly: we spent quite some time discussion (roughly) 5 different proof to the assertion that $\|A Q\|_2 = \|A\|_2$, where Q is unitary. These included

- Use that $\|A\|_2$ is the (positive) square root of the largest eigenvalue of $AA^\star$.
- Use that $\|Q\|_2 = 1$, and the consistency of matrix norms:
$$\begin{aligned}\|AQ\|_2 \leqslant \|A\|_2\|Q\|_2 = \|A\|_2 = \|AQQ^\star\|_2 \\ \leqslant \|AQ\|_2\|Q^\star\| = \|AQ\|_2.\end{aligned}$$
-
$$\begin{aligned}\|AQ\|_2 := \max_{\|x\|\neq 0} \frac{\|AQx\|_2}{\|x\|_2} = \max_{\|x\|\neq 0} \frac{\|AQx\|_2}{\|Qx\|_2} \\ = \max_{\|Qx\|\neq 0} \frac{\|AQx\|_2}{\|Qx\|_2} =: \|A\|_2.\end{aligned}$$
- Etc.

8.1 The (Full) SVD

The (full) SVD of $A \in \mathbb{C}^{m\times n}$ is the factorisation $A = U\Sigma V^\star$, where

$$U \in \mathbb{C}^{m\times m} \text{ is unitary,}$$

$$\Sigma \in \mathbb{R}^{m\times n} \text{ is diagonal,}$$

$$V \in \mathbb{C}^{n\times n} \text{ is unitary.}$$

Further, the diagonal entries of Σ are $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \sigma_p \geqslant 0$, where $p = \min(m, n)$.

The columns of U are called the *left singular values* of A, and the columns of V are called the *right singular values* of A.

We then started on the proof of the following result:

Theorem 8.1. *Every matrix has an SVD, and the singular values are uniquely determined. If the matrix is square and the singular values distinct, the left and right singular vectors are uniquely determined up to complex sign.*

In the class, we looked a (rushed) argument on the existence of the SVD. For the rest of the proof, have a look at Trefethen+Bau, Lecture 4.