9 Nine more theorems about the SVD

This is just a draft version of today's notes; a longer version with more exercises will be posted to the website soon...

9.1 The theorems

Theorem 9.1. *The rank of* A *is the number of nonzero singular values.*

Theorem 9.2. $\mathrm{range}(A) = \mathrm{span}(u_1, u_2, \dots, u_r)$ *and* $\mathrm{null}(A) = \mathrm{span}(v_{r+1}, \dots, v_n)$ *where* r *is the number of nonzero singular values of* A.

In (6.1), we defined the most important example of a matrix norm that is *not* induced by a vector norm – the *Frobenius norm*:

$$\|A\|_F := \Big(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\Big)^{1/2}.$$

Theorem 9.3. $\|A\|_2 = \sigma_1$. $\|A\|_F = \|\operatorname{diag}(\Sigma)\|_2$.

Theorem 9.4. *The singular values of* A *are the square roots of the eigenvalues of* $A^\star A$.

Theorem 9.5. *If* A *is hermitian, the singular values of* A *are the absolute values of the eigenvalues of* A.

Theorem 9.6. *If* $A \in \mathbb{C}^{m\times m}$, $|\det(A)| = \prod_{i=1}^{m} \sigma_i$.

Theorem 9.7. A *is the sum of the* r *rank-one matrices*

$$A = \sum_{j=1}^{r} \sigma_j u_j v_j^\star.$$

Theorem 9.8. *Let* A_V *be the rank-*ν *approximation to* A

$$A_\nu = \sum_{j=1}^{r} \sigma_j u_j v_j^\star.$$

Then

$$\|A - A_\nu\|_2 = \inf_{\mathrm{rank}(X)\leqslant \nu} \|A - X\|_2 = \sigma_{\nu+1},$$

where if $\nu = p = \min(m, n)$, *we define* $\sigma_{\nu+1} = 0$.

Theorem 9.9 (Schmidt)**.**

$$\|A - A_\nu\|_F = \inf_{\mathrm{rank}(X)\leqslant \nu} \|A - X\|_F.$$

All the above results emphasise the power the power of the SVD as an important theoretical and computational tool.

Computing the SVD is an important task, but not a trivial one. There are a few different approaches, but a key idea is the QR-*factorisation*.

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9.2 Exercises

Exercise 9.1. Suppose you want to solve the linear system $Ax = \mathbf{b}$, and you already have computed the SVD of A. How would you solve the linear system?

Exercise 9.2. Prove the second part of Theorem 9.2: if the SVD of A is $U\Sigma V^\star$, then the null space of A is spanned by columns $r+1, \dots, n$ of V, where where r is the number of nonzero singular values of A.

Exercise 9.3. Prove Theorem 9.9.