10 Solving Differential Equations

Today (after finishing off a left-over topic from the SVD) we'll take a detour to study numerical schemes for solving differential equations: finite differences (first) and finite element methods. Both of these approaches lead of linear systems of equations that must be solved, which is our main topic of interest.

10.1 Some basic DE theory

The particular ordinary differential equations that we are interested in are called *Boundary Value Problems (BVPs)*. We'll consider problems in one and two dimensions.

The one-dimensional problem is: *find a function* $u = u(x)$ *such that*

$$-u''(x)+a(x)u'(x)+b(x)u(x) = f(x), \quad \text{for } 0 \leqslant x \leqslant 1,$$

with $u(0) = u_0$ and $u(1) = u_1$. Here, there is no importance associated with posing the DE on the unit interval: we could easily transform to any interval. Also, it is easy to transform to a problem that satisfies homogeneous boundary conditions.

It is helpful to formulate this problem in terms of a *differential operator*. Later we'll see how to replace the differential equation with a matrix-vector problem, where the matrix approximates the differential operator. Our operator is:

$$L(u) := -u''(x) + a(x)u'(x) + b(x)(u). \tag{10.1}$$

Then the general form of our BVP is: *solve*

$$\begin{aligned} L(u) = f(x) \quad &\text{for} \quad 0 < x < 0, \\ u(0) = u_0, u(1) &= u_1. \end{aligned} \tag{10.2}$$

We make the assumptions that a, b and f are continuous and have as many continuous derivatives as we would like. Furthermore we always assume that there is a number β_0 such that $b(x) \geqslant \beta_0 > 0$.

Theorem 10.1 (Maximum Principle)**.** *Suppose* u *is a function such that* $Lu \geqslant 0$ *on* $(0, 1)$ *and* $u(0) \geqslant 0$, $u(1) \geqslant 0$*. Then* $u \geqslant 0$ *for all* $x \in [0, 1]$*.*

This result has numerous applications. For example,

Corollary 10.2. *Suppose that* $a \equiv 0$*. Define* $C = \max_{0 \leqslant x \leqslant 1} |f(0)|/\beta_0$*. Then* $u(x) \leqslant C$*.*

Corollary 10.3. *There is at most one solution to (10.2).*

10.2 Numerical solution of BVPs

Most boundary value problems are difficult or impossible to solve analytically. So we need approximations. Two of the most popular numerical methods for this are

- Finite Difference Methods (FDMs) — where one approximates the differential operator L, based on Taylor series expansions.
- Finite Element Methods (FEMs) – where to solution space is approximated.

We'll next study FDMs, then FEMs. Then we'll study numerical schemes for solving the linear systems that they give rise to.

10.3 Exercises

Exercise 10.1. Although (10.2) is expressed in terms of arbitrary boundary conditions, without loss of generality, one can assume that the differential equations has homogeneous boundary conditions. That is, that $u(x) = 0$ at the boundaries.

Show how to find a problem which is expressed in terms of the differential operator defined in (10.1), has homogeneous boundary conditions, and with a solution that differs from this one only by a known linear function.

Exercise 10.2. Prove Corollary 10.2.

Exercise 10.3. Prove Corollary 10.3.