11 Finite difference methods (1)

Choose a set of points $\{0 = x_0 < x_1 < \cdots < x_N = 0\}$, called the *mesh*, with $h = x_i - x_{i-1} = 1/N$. Replace the derivatives in (10.2) with difference formulae. We can deduce these in several ways:

(i) Geometrically. Recall, all we are really looking for is the slope of the tangent to $f(x)$ at $x = x_i$.

(ii) By finding a polynomial that interpolates f and differentiating that.

(iii) Undetermined coefficients.

(iv) Taylor's Theorem.

For more details, see, e.g., Lecture 24 of Stewart's *Afternotes* ([Ste98]). We'll focus on the Taylor Series approach.

Theorem 11.1 (Taylor's Theorem). *Suppose that* f *is continuous and* $n+1$ *times differentiable on the interval* $[a,b]$. *Then for every* $x \in [a,b]$, *we can write*

$$u(x) = u(a) + (x-a)u'(a) + \frac{(x-a)^2}{2}u''(a) + \frac{(x-a)^3}{3!}u'''(a) + \cdots + \frac{(x-a)^n}{n!}u^{(n)}(a) + R_n.$$

where the remainder R_n *is be given by*

$$R_n = \frac{(x-a)^{n+1}}{(n+1)!}u^{(n+1)}(\eta),$$

for some $\eta \in (a,b)$. *Q*

We can use this to deduce our rules. In particular:

- the backward difference scheme for u'
$$u'(x_i) = \frac{1}{h}(u(x_i) - u(x_{i-1})) + \frac{h}{2}u''(\tau), \quad (11.1)$$
- The central difference scheme for u'
$$u'(x_i) = \frac{1}{2h}(u(x_{i+1}) - u(x_{i-1})) + \mathcal{O}(h^2) \quad (11.2)$$
- The central difference scheme for u''.
$$u''(x_i) = \frac{1}{h^2}(u_{i-1} - 2u_i + u_{i+1}) + \mathcal{O}(h^2). \quad (11.3)$$

11.1 Discretisation

Let's restrict our interest, for now, to the DE:

$$-u'' + b(x)u(x) = f(x). \qquad (11.4)$$

We will seek an approximation on the uniformly spaced points $x_i = ih$ where $h = 1/N$. We denote by u_i the approximation for u at $x = x_i$. The finite difference scheme gives the following system of linear equations:

$$U_0 = 0,$$
$$-\left(\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2}\right) + b(x_i)U_i = f(x_i),$$
$$i = 1, 2, \dots, n-1,$$
$$U_n = 0.$$

In keeping with our approach to studying differential equations, we'll think of the finite difference formula as defining an operator:

$$L^N U_i := -\left(\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2}\right) + b(x_i)U_i. \quad (11.5)$$

Some of the properties of the differential operator carry over to the difference operator. In particular, we have the following analogue of Theorem 10.1.

Theorem 11.2 (Discrete Maximum Principle). *Suppose that* $\{V_i\}_{i=0}^N$ *is a mesh function such that* $L^N V_i \geqslant 0$ *on* $x_1, \dots x_{N-1}$, *and* $V_0 \geqslant 0$, $V_N \geqslant 0$. *Then* $V_i \geqslant 0$ *for* $i = 0, \dots, N$.

This very useful result can be used to establish many facts, including that the solution to (11.5) is unique. Another important result is:

Corollary 11.3. *Let* $\{V_i\}_{i=0}^N$ *be any mesh function with* $V_0 = V_N = 0$. *Then*

$$|V_i| \leqslant \beta_0^{-1} \|L^h V_i\|_\infty$$

11.2 Exercises

Exercise 11.1. Suppose that we actually want to solve

$$-\varepsilon u'' + a u'(x) = f(x) \quad \text{on } (0,1),$$

where ε and a are positive constants.

If the difference operator now includes the approximation for $u'(x_i)$ given in (11.1), show that L^N satisfies a discrete maximum principle.

If we use the approximation for $u'(x_i)$ given in (11.2), what assumptions must be made on a, ε and N for L^N to satisfy a discrete maximum principle?