13 Finite difference methods (2)

We actually spend most of the lecture reviewing material from Lecture 11, since it had been a week since we covered that, and proving Theorem 11.2 and Corollary 11.3.

13.1 Numerical Analysis in 1D

Let u be the solution to the differential equation (11.4), and let U be its finite difference approximation (11.5). We want to estimate the error: $\|u - U\|_\infty$.

Let δ^2 be the operator we derived in (11.3):

$$\delta^2 U_i := \frac{1}{h^2}\big(U_{i-1} - 2U_i + U_{i+1}\big),$$

and recall that we can deduce from truncated Taylor's series that

$$u''(x_i) - \delta^2 u(x_i) = \frac{h^2}{12} u^{(iv)}(\tau) \quad \text{for } \tau \in (x_{i-1}, x_{i+1}).$$

From this one can show that, for all i

$$|u''(x_i) - \delta^2 u(x_i)| \leqslant \frac{h^2}{12} M, \tag{13.1}$$

where $\|u^{(iv)}(x)\| \leqslant M$ for all x in $[0, 1]$.

We will use Theorem 11.2, Corollary 11.3, and ((13.1)) to show that

Theorem 13.1. *Suppose that* $u(x)$ *is the solution to the problem:*

$$Lu(x) = f(x), \qquad u(0) = u(1) = 0$$

and $\|u^{(iv)}(x)\|_\infty \leqslant M$. *Let* U *be the mesh function that solves*

$$L^N U_i = f(x_i) \quad \textit{for } i = 1, 2, \ldots, N-1, \quad U_0 = U_N = 1.$$

Then

$$\max_{i=0,\ldots,N} |u(x_i) - U_i| \leqslant \frac{h^2}{12}\frac{M}{\beta_0}$$

One might think that the numerical linear algebraist would not care too much about such analyses. However, we'll see that when we use iterative methods for solving linear systems that arise from the discretisation of DEs, we need to know the discretisation error, so it can be matched by the solver error.

13.2 Practical issues

Because our problem of interest has zero boundary conditions, the finite difference method is equivalent to solving a linear system of $N-1$ equations. That is, we think of L^N as a *tridiagonal* matrix. Most of the next few lectures will be concerned with solving the associated linear system.

13.3 Exercises

Exercise 13.1 (Demmel, Lemma 6.1)**.** Suppose we want to apply a finite difference method to the problem

$$-u''(x) = f(x) \quad \text{on } (0, 1)$$

with $u(0) = u(1) = 0$, and we decide to write the linear system as

$$T\mathbf{U} = h^2 \mathbf{f},$$

where $\mathbf{U} = (U_1, \ldots, U_{N-1})^T$ and $\mathbf{f} = (f(x_1), \ldots, f(x_{N-1})$. Show that the eigenvalues of T are

$$\lambda_j = 2\big(1 - \cos(\pi j/N)\big),$$

with corresponding normalised eigenvectors

$$z_i^j = \sqrt{\frac{2}{N}} \sin(j i \pi/N).$$

Exercise 13.2. Write (11.5) as a matrix-vector equation. Show that the eigenvalues of the matrix are positive real numbers. (Note: unlike Exercise , we have that $b(x) \geqslant \beta_0 > 0$) *Hint: read up on Gerschgorin's theorems.*