14 Finite differences in 2D

14.1 A two dimensional differential equation

The next problem we wish to solve is

$$-\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big)u(x,y)+b(x,y)=f(x,y), \qquad (14.1)$$

for all x in the unit square: $(0,1)\times(0,1)$, and with the homogeneous boundary conditions:

$$u(0,y)=u(1,y)=u(x,0)=u(x,1)=0.$$

Often we express this in terms of the Laplacian operator:

$$\Delta u := \Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big)u.$$

For simplicity, initially we'll consider $b\equiv 0$ in (14.1), which gives the standard *Poisson* problem:

$$-\Delta u(x,y)=f(x,y) \quad \text{ on } (0,1)^2.$$

14.2 Finite differences in two dimensions

[You might need to notes from class to make complete sense of this section. You should also read Section 6.3.2 of Demmel's *Applied Numeral Linear Algebra* [Dem97]].

We divide the unit square into a regular grid of $(N+1)^2$ points, with $h=x_i-x_{i-1}=y_j-y_{j-1}$. Let us denote the approximation for u at the point (x_i,v_j) by $U_{i,j}$. We can approximate the partial derivatives using the difference formula of (11.3):

$$\frac{\partial^2}{\partial x^2}u(x_i,y_j)\approx\frac{1}{h^2}(U_{i-1,j}-2U_{i,j}+U_{i+1,j}).$$

$$\frac{\partial^2}{\partial y^2}u(x_i,y_j)\approx\frac{1}{h^2}(U_{i,j-1}-2U_{i,j}+U_{i,j+1}).$$

Combining these we get

$$\begin{pmatrix} & -U_{i,j+1} & \\ -U_{i-1,j} & 4U_{i,j} & -U_{i+1,j} \\ & -U_{i,j-1} & \end{pmatrix} = h^2 f(x_i,y_j). \qquad (14.2)$$

There are two ways of representing (14.2):

(a) we can think of U as a matrix of values, which is what we'll do for the rest of this lecture, or

(b) with a suitable ordering, we can think of U as a vector, which is what we'll do for most of the rest of this module.

14.3 The matrix representation of (14.2)

There are a total of $(N+1)^2$ values of $U_{i,j}$ we need to compute, each associated with the a node on the two-dimensional mesh (see diagram from class). However, we know the values at the boundary, so we only need to compute $(N-1)^2$ values:

$$U_{i,j} \quad i=1,\dots,N-1, \quad j=1,\dots,N-1.$$

Suppose we store these values as a matrix

$$U=\begin{pmatrix} U_{1,1} & U_{1,2} & \cdots & U_{1,N-1}. \\ U_{2,1} & U_{2,2} & \cdots & U_{2,N-1}. \\ \vdots & \vdots & & \vdots \\ U_{N-1,1} & U_{N-1,2} & \cdots & U_{N-1,N-1}. \end{pmatrix}$$

Let $T_N\in\mathbb{R}^{(N-1)\times(N-1)}$ matrix:

$$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}.$$

(Recall also Exercise 13.1.)

The we saw that we can represent the method (14.2) as

$$(T_N U + U T_N) = h^2 f(x_i,y_j). \qquad (14.3)$$

We finished by observing that if, T_N has eigenvalues λ_i and λ_j with corresponding eigenvectors z_i and z_j, then $\lambda_i+\lambda_j$ is an "eigenvalue" of the system represented in (14.3), with corresponding eigenvector $V=z_i z_j^T$, in the sense that

$$T_N V + V T_N = (\lambda_i+\lambda_j)V.$$

The argument presented was taken directly from [Dem97, p. 271].