15 Properties of the 2D finite difference matrix

15.1 Recall...

We began by recapping on Lecture 14: we wish to find a numerical approximation to the solution to the Poisson Problem:

$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u(x,y) = f(x,y),$$

for all $(x,y) \in (0,1)^2$, and $u = 0$ on the boundary of $(0,1)^2$. We construct a mesh (i.e., a grid) of evenly spaced points $\{(x_i, y_j)\}_{i=0,j=0}^{N,N}$, where $x_i - x_{i-1} = y_j - y_{j-1} = h$.

We revisited that, if we think of the $U_{i,j}$ as being represented by an matrix, then (14.2) could be expressed as et $T_N \in \mathbb{R}^{(N-1)\times(N-1)}$ matrix:

$$(T_N U + U T_N) = h^2 f(x_i, y_j).$$

where T_N is

$$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}. \qquad (15.1)$$

15.2 The matrix version

We now want to see how to express the method as a more standard matrix-vector equation.

For the $(N+1)\times(N+1)$ mesh we described, we are actually only interested in the $(N-1)^2$ interior points, since we know the values at the boundary. With lexicographic ordering, we represent U as

$$U = (U_1, U_2, \dots, U_{(N-1)^2})^T,$$

where U_k corresponds to $U_{i,j}$ if $k = i + (N-1)(j-1)$. The k^{th} row of the method (14.2) becomes

$$\begin{pmatrix} & -U_{k+(N-1)} & \\ -U_{k-1} & 4U_k & -U_{k+1} \\ & -U_{k-(N-1)} & \end{pmatrix}$$

15.3 Example: N=3

See you notes from class, which included showing that if we write the system as

$$T_{3,3}\mathbf{U} = h^2\mathbf{F},$$

then $T_{3,3}$ is the matrix

$$\left(\begin{array}{ccc|ccc|ccc} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{array}\right)$$

which can be written more succinctly, and more generally, as

$$\left(\begin{array}{c|c|c} T_N + 2I & -I & 0 \\ \hline -I & T_N + 2I & -I \\ \hline 0 & -I & T_N + 2I \end{array}\right)$$

15.4 Kronecker products

Definition 15.1. If $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$, then the *Kronecker product* of A and B is the $A \otimes B \in \mathbb{R}^{(mp)\times(nq)}$ given by

$$\begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & & & \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{pmatrix}$$

We then did several examples, culminating in noting that

$$T_{3,3} = I \otimes T_3 + T_3 \otimes I.$$

To show that this holds in general, we introduce a new operator:

Definition 15.2. If $A \in \mathbb{R}^{m\times n}$ then $\text{vec}(A)$ is the vector in $\mathbb{R}^{mn}$ formed from stacking the columns of A. That is

$$\text{vec}(A) = (a_{11}, a_{21}, a_{31}, \dots, a_{m1}, a_{21}, \dots, a_{mn})^T.$$

Theorem 15.3 (Demmel, Lemma 6.2)**.** *Let* $A, B, X \in \mathbb{R}^{n\times n}$*. Then*

(a) $(AX) = (I \otimes A)\text{vec}(X)$.

(b) $(XB) = (B^T \otimes I)\text{vec}(X)$.

(c) The formulation for the finite difference method

$$(T_N U + U T_N) = h^2 F,$$

is equivalent to

$$T_{N,N}\text{vec}(U) = (I \otimes T_N + T_N \otimes I)\text{vec}(U) = h^2\text{vec}(F).$$

We left (a) and (b) as exercises. The proof of (c) comes directly from (a) and (b) and from noting that T_N is symmetric.

15.5 Exercise

Exercise 15.1. Prove Parts (a) and (b) of Theorem 15.3.