17 LU-factorisation

17.1 Formulae for L and U

In Lecture 16, we saw that applying Guassian Elimination to a linear system $A\mathbf{x} = \mathbf{b}$ is equivalent to left-multiplication of A by series unit lower triangular matrices, yielding an upper triangular matrix.

Definition 17.1. The **LU-factorization** of the matrix is a unit lower triangular matrix L and an upper triangular matrix U such that $LU = A$.

Now we'll derive a formula for L and U. Since $A = LU$, the entries of A are $a_{i,j} = (LU)_{ij} = \sum_{k=1}^{m} l_{ik}u_{kj}$. Since L and U are triangular,

$$\text{If } i \leqslant j \quad \text{then} \quad a_{i,j} = \sum_{k=1}^{i} l_{ik}u_{kj}$$

This can be written as

$$a_{i,j} = \sum_{k=1}^{i-1} l_{ik}u_{kj} + l_{ii}u_{ij}.$$

But $l_{ii} = 1$ so:

$$u_{i,j} = a_{ij} - \sum_{k=1}^{i-1} l_{ik}u_{kj} \qquad \begin{cases} i = 1, \dots, j-1, \\ j = 2, \dots, m. \end{cases} \tag{17.3a}$$

Similarly, one can show that

$$l_{i,j} = \frac{1}{u_{jj}}\left(a_{ij} - \sum_{k=1}^{j-1} l_{ik}u_{kj}\right) \qquad \begin{cases} i = 2, \dots, m, \\ j = 1, \dots, i-1. \end{cases} \tag{17.3b}$$

17.2 Solving $LU\mathbf{x} = \mathbf{b}$

We can now factorise A as $A = LU$. But we are trying to solve the problem: *find* $\mathbf{x} \in \mathbb{R}^m$ *such that* $A\mathbf{x} = \mathbf{b}$, *for some* $\mathbf{b} \in \mathbb{R}^m$. So solve

$$L\mathbf{y} = \mathbf{b} \text{ for } \mathbf{y} \in \mathbb{R}^n \quad \text{and then } U\mathbf{x} = \mathbf{y}.$$

Because L and U are triangular, this is easy.

Example 17.2. Use LU-factorisation to solve

$$\begin{pmatrix} -1 & 0 & 1 & 2 \\ -2 & -2 & 1 & 4 \\ -3 & -4 & -2 & 4 \\ -4 & -6 & -5 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ -1 \\ 1 \end{pmatrix}$$

17.3 Pivoting

Not every matrix has an LU-factorisation. Consider, for example

$$A = \begin{pmatrix} 0 & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \text{and } B = \begin{pmatrix} d & e & f \\ 0 & b & c \\ g & h & i \end{pmatrix}$$

A does not have an LU-factorisation, but B does. We can think of B as a permutation of A.

Definition 17.3. P is a *Permutation Matrix* if every entry is either 0 or 1 (i.e., it is a Boolean Matrix) and if all the row and column sums are 1.

Theorem 17.4. *For any* $A \in \mathbb{R}^{m\times m}$ *there exists a permutation matrix* P *such that* $PA = LU$.

We won't prove this in class, but you can find the argument in any good numerical analysis textbook.

17.4 The computational cost

How many computational steps (additions and multiplications) are required to compute the LU-factorisation of A? Suppose we want to compute $l_{i,j}$. From the formula (17.3b) we see that this would take j—2 additions, $j-1$ multiplications, 1 subtraction and 1 division: a total of $2j - 2$ operations. Recall that

$$\sum_{i=1}^{k} i = \frac{1}{2}k(k+1), \text{ and } \sum_{i=1}^{k} i^2 == \frac{1}{6}k(k+1)(2k+1)$$

So the total number of operations required for computing L is

$$\sum_{i=2}^{m}\sum_{j=1}^{i-1}(2j-1) = \sum_{i=2}^{m} i^2 - 2i + 1 =$$
$$\frac{1}{6}m(m+1)(2m+1) - m(m+1) \leqslant Cm^3$$

for some C. A similar (slightly smaller) number of operations is required for computing U.

17.5 Exercises

Exercise 17.1. Many textbooks and computing systems compute the factorisation $A = LDU$ where L and U are unit lower and *unit* upper triangular matrices respectively, and D is a diagonal matrix. Show that such a factorisation exists.

Exercise 17.2. Suppose that $A \in \mathbb{R}^{m\times m}$ is nonsingular and symmetric, and that every leading principle submatrix of A is nonsingular. Use Exercise 17.1 so show that A can be factorised as $A = LDL^T$.
How would this factorization be used to solve $A\mathbf{x} = \mathbf{b}$?

Exercise 17.3. Consider the matrix

$$H_3 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}.$$

Write down the LU-factorization of H_3, and hence solve the linear system $H_3\mathbf{x} = b$ where $b = \frac{1}{12}(6, 2, 1)^T$.