18 Symmetric positive definiteness

These notes were revised from the ones provided in class. In particular, the exercises were added.

18.1 Overview

For the next few classes, we want to study methods for solving $Ax = b$ where A has special properties, such as being symmetric, sparse, or banded (all properties of the finite difference matrices from Lecture 15).

Suppose we wanted to solve $Ax = b$ where A is symmetric. Since $A = A^T$, we might like to think that we could use a version of LU-factorisation where $L = U^T$. However (see [Ips09, Section 3.6]), this can't be done. Consider the example

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

18.2 s.p.d. matrices

LU-factorisation is the most standard method. But many problems, such as those from finite difference methods, have very special structure that we can exploit in order to save time and memory. The first case we'll consider is if A is s.p.d.

Definition 18.1 (s.p.d.). A matrix, $A \in \mathbb{R}^{m\times m}$, is *symmetric positive definite* (s.p.d.) if and only if $A = A^T$ and $x^T A x > 0$ for all vectors $x \neq 0$.

It is easy to see that such a matrix is nonsingular. If A were s.p.d. and singular, then there would exist some vector x such that $Ax = 0$, and so $x^T A x = 0$. This is not possible since $x^T A x > 0$ for all x.

Theorem 18.2 (Demmel, Prop 2.2). *(a) If X^{-1} exists, A is s.p.d. $\iff$ $X^T A X$ is s.p.d.*

(b) If A is s.p.d., then any principle submatrix of A is s.p.d.

(c) A is s.p.d. $\iff$ $A = A^T$ and all the eigenvalues of A are positive.

(d) If A is s.p.d., then $a_{ii} > 0$ for all i, and $\max_{ij} |a_{ij}| = \max_i a_{ii}$.

(e) A is s.p.d. $\iff$ there exists a unique lower triangular matrix L with positive diagonal entries such that $A = LL^T$. This is called the Cholesky factorisation of A.

In class we stepped through the proofs of (a)–(d). We'll save the proof of Theorem 18.2-(e) until the next lecture.

18.3 Exercises

Exercise 18.1. Give (at least) 3 different proofs that an s.p.d. matrix is nonsingular.

Exercise 18.2. So now we know that if A is s.p.d., then A^{-1} must exist. Must A^{-1} be s.p.d?

Exercise 18.3 (Stephen R)**.** Give (at least) 2 different proofs that the determinant of an s.p.d. matrix is strictly positive.

Exercise 18.4. We say a matrix $A \in \mathbb{C}^{m\times m}$ is *hermitian positive definite* if $z^\star A z > 0$ for all $z \in \mathbb{C}^m$. Which of the five properties in Theorem 18.2 extend to *hermitian positive definite* matrices?

Exercise 18.5 (James McT)**.** Is it possible for a matrix $A \in \mathbb{R}^{m\times m}$ to be positive definite, in the sense that $x^T A x > 0$ for all x, and yet not be symmetric? (Hint: take $A \in \mathbb{R}^{2\times 2}$ to be of the form

$$A = \begin{pmatrix} 0 & b \\ c & 1 \end{pmatrix},$$

and try to find b and c such that $x^T A x > 0$).

Exercise 18.6 (Thái)**.** In Theorem 18.2-(d) we show that the maximum entry always occurs on the diagonal. Must the matrix always be diagonally dominant? Prove this, or give a counter-example.

Exercise 18.7. [Ips09, p66–67] Given a matrix A partitioned into four submatrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

a *Schur compliment* of A is $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$. In *Fact 3.27* of Ilse Ipsen's *Numerical Matrix Analysis* (available at http://www4.ncsu.edu/~ipsen/) it is shown that if A is s.p.d., so too is the Schur compliments.

Use this to show that, if A is s.p.d., then $|a_{ij}| \leqslant \sqrt{a_{ii}a_{jj}}$ for $i \neq j$. Show further that $|a_{ij}| \leqslant (a_{ii} + a_{jj})/2$ for $i \neq j$.