19 Cholesky Factorisation

(*As is often the case, these notes were heavily revised after the class, and so differ greatly from the version handed out in class*).

19.1 Recall: s.p.d. matrices

At the end of the last class, we had proved Parts (a)–(d) of Theorem 18.2. We'll start with a few examples of s.p.d. matrices.

The matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2. \end{pmatrix}$$

is s.p.d. We can deduce this from the celebrated *Gerschgorin theorems.*

19.2 Gerschgorin's First Theorem

(See, e.g., [Dem97, Thm. 2.9], [Saa03, 4.6]). Given a matrix $A \in \mathbb{R}^{m\times m}$, the *Gerschgorin Discs* D_i are the m discs in the complex plane with centre a_{ii} and radius r_i:

$$r_i = \sum_{j=1, j\neq i}^{m} |a_{ij}|.$$

So $D_i = \{z \in \mathbb{C} : |a_{ii} - z| \leqslant r_i\}$. It can then be shown (Gerschgorin's First Theorem) that all the eigenvalues of A are contained in the union of the Gerschgorin discs. Furthermore (Gerschgorin's Second Theorem) if k of discs are disjoint (have an empty intersection) from the others, their union contains k eigenvalues. [1].

This has many applications. For example, we can now see that the eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2. \end{pmatrix}$$

are contained in the interval $[1, 2]$, so they must be positive. By Theorem 18.2-(c). it must be s.p.d. See also Exercise 19.2.

We then observed that T_N from (15.1) is s.p.d. This is true since, by the Gerschgorin results, the eigenvalues are in the interval $[0, 2]$. Next we apply row reduction to show that T_N is invertable, and thus that the eigenvalues are actually in $(0, 2]$. So they are strictly positive. (The way I'd planned to do this in case was the invoke the formula in Exercise 13.1).

Example 19.1. Can we show that

$$T_{N,N} = I \otimes T_N + T_N \otimes I,$$

is s.p.d?

[1] In class, I'd initially, and incorrectly, assumed everyone knew these theorems. We'll devote a whole lecture to them some time soon

Answer: Recall from Section 14.3 (but see also [Dem97, p. 271]), that every eigenvalue of $T_{N,N}$ is the sum of two eigenvalues of T_N. Since the eigenvalues of T_N are positive, so too are those of $T_{N,N}$.

19.3 Another aside

We also had a short detour to discuss

- A is *diagonal* if its only non-zero entries are found on the main diagonal. That is $a_{ij} = 0$ if $i \neq j$.
- A is *tridiagonal* if its only non-zero entries are found on the main diagonal, the subdiagonal or the super-diagonal. That is $a_{ij} = 0$ if $|i - j| > 1$.
- A is *bidiagonal* if its only non-zero entries are found on the main diagonal, and one of the sub-diagonal or the super-diagonal (but not both). That is A is triangular and tridiagonal.

19.4 Part (e) of Theorem 18.2 (finally!)

We now consider Part (e) of Theorem 18.2: $A \in \mathbb{R}^{m\times m}$ is symmetric positive definite (s.p.d.) if and only of there exists a unique lower triangular matrix L with positive diagonal entries such that $A = LL^T$. *This is called the Cholesky factorisation of* A.

It is important to note that this is an "if and only if" statement. For example, a standard way of testing if a matrix is s.p.d. is to try to compute the Cholesky factorisation, and to see if it fails.

19.5 Exercises

Exercise 19.1. [Dem97, Exer 6.2] Derive a formula for the LU factorisation of T_N that is similar to the Cholesky factorisation in (20.1).

Exercise 19.2. Recall that a matrix $A \in \mathbb{R}^{m\times m}$ is (strictly) diagonally dominant if

$$a_{ii} > \sum_{j\neq i} |a_{ij}A| \quad \text{for } i = 1, \dots, m.$$

Show that a symmetric diagonally dominant matrix must be s.p.d.

Exercise 19.3. Show that the converse of Exercise 19.2 does not hold. That is, show (by example) that it is possible for a matrix to be s.p.d. but not diagonally dominant.