23 The graph of a matrix

23.1 Graph of a matrix

Recall that a *graph* $G = (V, E)$ consists of a set of vertices (nodes) $V = \{1, \ldots, n\}$ and edges $E = \{(i, j) | i, j \in V\}$. We visualise this as nodes V with a line (edge) joining i to j if $(i, j) \in E$, with an arrow indicating that this edge is *from* i *to* j.

An *undirected* graph is one where the edges don't have arrows, but really this is just a short hand way of depicting that if there is a (directed) edge from i to j, there is also one from j to i.

We say there is a *path* from v_0 to v_n if there is a sequence of nodes $(v_0, v_1, \ldots, v_n)$ such that the edges (v_0, v_1), (v_1, v_2), dots (v_{n-1}, v_n) are all in E. We write this as $v_0 \rightsquigarrow v_n$.

Given an $m \times m$ matrix A, the *graph of the matrix* $G(A)$ is the graph with m vertices such that there is an edge (i, j) if $a_{i,j} \neq 0$.

Example 23.1. The graph of the one-dimensional Laplacian is ...

Example 23.2. Let A be the matrix

$$\begin{pmatrix} 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 4 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 \end{pmatrix}$$

Its graph is shown below in Figure 23.1

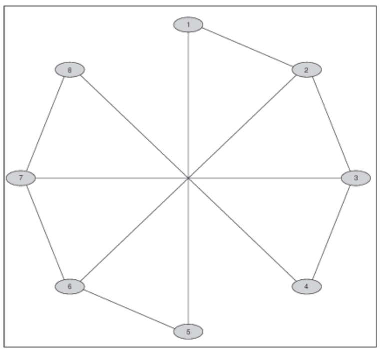

Figure 23.1: The graph of the matrix in Example 23.2

23.2 Fill-in

Recall that if $A = LL^T$, the *fill-in* refers to entries of $L + L^T$ that do not appear in A.

The graph $G(L + L^T)$ is called the *filled graph of* A.

Theorem 23.3 (Thm. 4.1 of [Dav06])**.** *The edge* (i, j) *is in the (undirected) graph* $G(L + L^T)$ *if and only if there exists a path* $i \rightsquigarrow j$ *in* $G(A)$ *where all nodes in the path, except* i *and* j *are numbered less than* $\min(i, j)$.

To see what this is telling us, consider the Cholesky factorisation of the matrix on Example 23.2. Here we just show the sparsity pattern.

$$\begin{pmatrix} * & & & & & & & \\ * & * & & & & & & \\ & * & * & & & & & \\ & & * & * & & & & \\ * & * & * & * & * & & & \\ & * & * & * & * & * & & \\ & & * & * & * & * & * & \\ & & & * & * & * & * & * \end{pmatrix}$$

23.3 Characterising fill-in for Cholesky

We need to distinguish between zeros that appear in L that are entirely due to the structure of A, and those that appear due to cancellation.

Example 23.4. If $A = I$, then all off-diagonal entries of L are zero, by necessity. However, if

$$A = \begin{pmatrix} 4 & -1 & -4 \\ -1 & 7 & 1 \\ -4 & 1 & 8 \end{pmatrix} \text{ then } L = \begin{pmatrix} 2 & & \\ -1\frac{1}{2} & \frac{3\sqrt{3}}{2} & \\ -2 & 0 & 2 \end{pmatrix}.$$

That $l_{32} = 0$ is due to cancellation.

Theorem 23.5 (Thm. 4.2 of [Dav06])**.** *If* $A = LL^T$*, and neglecting cancellation, if* $a_{ij} \neq 0$ *then* $l_{ij} \neq 0$.

Theorem 23.6 (Thm. 4.3 of [Dav06])**.** *If* $A = LL^T$*, and* $l_{ji} \neq 0$ *and* $l_{kj} \neq 0$ *where* $i < j < k$*, then* $l_{kj} \neq 0$.

23.4 Exercise

Exercise 23.1. Verify that L given in Example 23.4 is indeed the Cholesky factor of A.