25 Analysis of basic iterative methods

[Post-lecture version. This has minor changes from the one given out before class, including that equation numbers have been corrected.]

In Lecture 24 we introduced the Jacobi and Gauss-Seidel methods for solving linear systems iterativly. They work as follows: to find an approximate solution to the linear system

$$Ax = \mathbf{b},$$

we write the matrix A as $A = D - E - F$, where D is a diagonal matrix, E is strictly lower triangular, and F is strictly upper triangular. We choose an initial guess $x^{(0)}$, and then set

Jacobi:

$$\mathbf{x}^{(k+1)} = D^{-1}(E + F)\mathbf{x}^{(k)} + D^{-1}b \tag{25.1a}$$

Gauss-Seidel:

$$\mathbf{x}^{(k+1)} = (D - E)^{-1}F\mathbf{x}^{(k)} + (D - E)^{-1}b. \tag{25.1b}$$

Under certain conditions, these iterations will *converge* in the sense that $\|\mathbf{x} - \mathbf{x}^{(k)}\| \to 0$ as $k \to \infty$. That is the topic we want to study today. For more details, see [Saa03, Chap 4]).

25.1 General iteration

The Jacobi, Gauss-Seidel and Successive Overrelaxation (see Exercise 25.1) methods can be written as schemes of the form

$$\mathbf{x}^{(k+1)} = G\mathbf{x}^{(k)} + f, \tag{25.2}$$

where $G_J = I - D^{-1}A$, and $G_{GS} = I - (D - E)^{-1}A$.

Another useful formulation is express A by the splitting *A=M-N*, and the iteration as

$$\mathbf{x}^{(k+1)} = M^{-1}N\mathbf{x}^{(k)} + M^{-1}\mathbf{b}. \tag{25.3}$$

For example, for Jacobi's method $M = D$, and Gauss-Seidel, $M = D - E$.

Both (25.2) and (25.3) have natural interpretations in the limiting case (i.e., as $k \to \infty$).

25.2 Convergence?

We have to ask three questions about our basic iterative methods:

(a) will they converge to the correct solution?

(b) (when) will they converge at all?

(c) how quickly do they converge?

It is easy to see that, if the Jacobi and Gauss-Seidel methods converge, they do so to the solution of $A\mathbf{x} = \mathbf{b}$.

The answer Question (b) we'll introduce some notation.

Definition 25.1. The *spectrum* of a matrix A is the set of its eigenvalues. We write it $\mu(A)$. The *spectral radius* of a matrix is the modulus of its largest eigenvalue:

$$\rho(A) = \max_{\lambda \in \mu(A)} |\lambda|.$$

Theorem 25.2 (Thm 1.10 of [Saa03])**.** *The sequence* G^k*,* $k = 0, 1, \ldots$ *converges to zero if and only if* $\rho(G) < 1$.

We'll be a little lazy and in class, only consider the case where the matrix G is *diagonalizable.* For the general case, we should study the *Jordan canonical form.*

We can now use this to show that the sequence generated by (25.2) converges providing that $\rho(G) < 1$.

Example 25.3. If the matrix A in the system $A\mathbf{x} = \mathbf{b}$ is strictly diagonally dominant, then the associated Jacobi iteration will converge.

In general, it is not so easy to work out $\rho(G)$ for a given matrix G. However, since $\rho(G) \leqslant \|G\|$ for any consistent matrix norm (see Exercise 26.2), then we can take $\|G\|$ is a sufficient (but not necessary) condition for convergence.

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We can now tell when one of our methods will converge. The next question we'll deal with is *how fast* convergence occurs.

25.3 Exercises

Exercise 25.1. Find out what the *Successive Overrelaxation* (SOR) method is. Show how to write it in the form of (25.1). Furthermore, find the formula for M and N if SOR it expressed in the form (25.3).

Exercise 25.2. The on-line notes for Lecture 24 are accompanied by a Matlab programme that generated Tables 24.1 and 24.2. Write a similar Matlab programme that implements SOR.